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A PROXIMAL POINT ALGORITHM FOR ZEROS OF MONOTONE OPERATORS

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Abstract. In this article, a proximal point algorithm is investigated for treating zeros of maximal monotone mappings. Strong convergence theorems are established in the framework of Hilbert spaces.

Keywords: monotone operator; nonexpansive mapping; fixed point; iterative process.

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1. Introduction-Preliminaries

Recently, algorithms have been studied as an effective and powerful tool for studying a wide class of real world problems which arise in economics, finance, and network; see [1-9] and the references therein.

Throughout this paper, we assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let T be a set-valued mapping.

(a) The set $D(T)$ defined by

$$D(T) = \{u \in H : T(u) \neq \emptyset\}$$

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is called the effective domain of T;

(b) The set $R(T)$ defined by

$$R(T) = \bigcup_{u \in H} T(u)$$

is called the range of T;

(c) The set $G(T)$ defined by

$$G(T) = \{(u,v) \in H \times H : u \in D(T), v \in R(T)\}$$

is said to be the graph of T.

Recall the following definitions.

(c) T is said to be monotone if

$$\langle u - v, x - y \rangle \geq 0, \quad \forall (u,x),(v,y) \in G(T);$$

(d) T is said to be maximal monotone if it is not properly contained in any other monotone operator.

For a maximal monotone $T : D(T) \to 2^H$, we can defined the resolvent of T by $J_t = (I + tT)^{-1}, \quad t > 0$. It is well known that $J_t : H \to D(T)$ is nonexpansive and $F(J_t) = T^{-1}(0)$, where $F(J_t)$ denotes the set of fixed points of J_t. The Yosida approximation T_t is defined by $T_t = \frac{1}{t}(I - J_t), \quad t > 0$. It is well known that $T_t x \in TJ_t x$, $\forall x \in H$ and $\|T_t x\| \leq |Tx|$, where $|Tx| = \inf\{\|y\| : y \in Tx\}$, for all $x \in D(T)$.

Let C be a nonempty, closed and convex subset of H. Next, we always assume that $T : C \to 2^H$ is a maximal monotone mapping with $T^{-1}(0) \neq \emptyset$, where $T^{-1}(0)$ denotes the set of zeros of T. The class of monotone mappings is one of the most important classes of mappings among nonlinear mappings. Within the past several decades, many authors have been devoting to the studies on the existence and convergence of zero points for maximal monotone mappings, see [10-18] and the references therein. A classical method to solve the following set-valued equation $0 \in Tx$, is the proximal point algorithm. To be more precise, start with any point $x_0 \in H$, and update x_{n+1} iteratively conforming to the following recursion $x_n \in x_{n+1} + \beta_n T x_{n+1}, n \geq 0$, where $\{\beta_n\} \subset [\beta, \infty)$, $(\beta > 0)$ is a sequence of real numbers.

In 1976, Rockafellar [19] gave an inexact variant of the method

$$x_0 \in H, \quad x_n + e_{n+1} \in x_{n+1} + \lambda_n T x_{n+1}, \quad n \geq 0,$$

where $\{e_n\}$ is regarded as an error sequence. This an inexact proximal point algorithm. It was shown that, if $\sum_{n=0}^{\infty} \|e_n\| < \infty$, then the sequence $\{x_n\}$ converges weakly to a zero of T provided that $T^{-1}(0) \neq \emptyset$. In [16], Güler obtained an example to show that Rockafellar's proximal point algorithm does not converge strongly, in general.

Recently, many authors studied the problems of modifying Rockafellar's proximal point algorithm so that strong convergence is guaranteed. Cho, Kang and Zhou [10] proved the following result.

Theorem CKZ. *Let H be a real Hilbert space, Ω a nonempty closed convex subset of H, and $T : \Omega \to 2^H$ a maximal monotone operator with $T^{-1}(0) \neq \emptyset$. Let P_Ω be the metric projection of H onto Ω. Suppose that, for any given $x_n \in H$, $\beta_n > 0$ and $e_n \in H$, there exists $\bar{x}_n \in \Omega$ conforming to the SVME (1.5), where $\{\beta_n\} \subset (0, +\infty)$ with $\beta_n \to \infty$ as $n \to \infty$ and $\sum_{n=1}^{\infty} \|e_n\|^2 < \infty$. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ such that*

(i) $\alpha_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any fixed $u \in \Omega$, define the sequence $\{x_n\}$ iteratively as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) P_\Omega(\bar{x}_n - e_n), \quad n \geq 0.$$

Then $\{x_n\}$ converges strongly to a fixed point z of T, where $z = \lim_{t \to \infty} J_t u$.

Lemma 1.1. [20] *Let H be a Hilbert space and C a nonempty, closed and convex subset H. For all $u \in C$, $\lim_{t \to \infty} J_t u$ exists and it is the point of $T^{-1}(0)$ nearest u.*

Lemma 1.2 [12] *For any given $x_n \in C$, $\lambda_n > 0$, and $e_n \in H$, there exists $\bar{x}_n \in C$ conforming to the following set-valued mapping equation (in short, SVME):*

$$x_n + e_n \in \bar{x}_n + \lambda_n T \bar{x}_n.$$

Furthermore, for any $p \in T^{-1}(0)$, *we have*

$$\langle x_n - \bar{x}_n, x_n - \bar{x}_n + e_n \rangle \leq \langle x_n - p, x_n - \bar{x}_n + e_n \rangle$$

and

$$\|\bar{x}_n - e_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - \bar{x}_n\|^2 + \|e_n\|^2.$$

Lemma 1.3 *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main results

Theorem 2.1. *Let* H *be a real Hilbert space,* C *a nonempty, closed and convex subset of* H *and* $T : C \to 2^H$ *a maximal monotone operator with* $T^{-1}(0) \neq \emptyset$. *Let* P_C *be a metric projection from* H *onto* C. *For any* $x_n \in H$ *and* $\lambda_n > 0$, *find* $\bar{x}_n \in C$ *and* $e_n \in H$ *conforming to the SVME, where* $\{\lambda_n\} \subset (0,\infty)$ *with* $\lambda_n \to \infty$ *as* $n \to \infty$ *and* $\|e_n\| \leq \eta_n \|x_n - \bar{x}_n\|$ *with* $\sup_{n\geq 0} \eta_n = \eta < 1$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0,1]$ *satisfying* $\alpha_n + \beta_n < 1$ *and the following control conditions:*

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0 \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Let $\{x_n\}$ *be a sequence generated by the following manner:*

$$x_0 \in H, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) P_C(\bar{x}_n - e_n). \quad n \geq 0, \tag{2.1}$$

where $f : C \to C$ *is a fixed contractive mapping. Then the sequence* $\{x_n\}$ *generated by (2.1) converges strongly to a zero point* z *of* T, *where* $z = \lim_{t\to\infty} J_t f(z)$, *if and only if* $e_n \to 0$ *as* $n \to \infty$.

Proof. First, show that the necessity. Assume that $x_n \to z$ as $n \to \infty$, where $z \in T^{-1}(0)$. It follows from (1.5) that

$$\begin{aligned}\|\bar{x}_n - z\| &\le \|x_n - z\| + \|e_n\| \\ &\le \|x_n - z\| + \eta_n \|x_n - \bar{x}_n\| \\ &\le (1+\eta_n)\|x_n - z\| + \eta_n\|\bar{x}_n - z\|.\end{aligned}$$

This implies that $\|\bar{x}_n - z\| \le \frac{1+\eta_n}{1-\eta_n}\|x_n - z\|$. It follows that $\bar{x}_n \to z$ as $n \to \infty$. Note that $\|e_n\| \le \eta_n\|x_n - \bar{x}_n\| \le \eta_n(\|x_n - z\| + \|z - \bar{x}_n\|)$. This shows that $e_n \to 0$ as $n \to \infty$.

Next, we show the sufficiency. From the assumption, we see $\|e_n\| \le \|x_n - \bar{x}_n\|$. For any $p \in T^{-1}(0)$. It follows from Lemma 1.2 that

$$\begin{aligned}\|P_C(\bar{x}_n - e_n) - p\|^2 &\le \|\bar{x}_n - e_n - p\|^2 \\ &\le \|x_n - p\|^2 - \|x_n - \bar{x}_n\|^2 + \|e_n\|^2 \\ &\le \|x_n - p\|^2.\end{aligned}$$

That is, $\|P_C(\bar{x}_n - e_n) - p\| \le \|x_n - p\|$. It follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + (1-\alpha_n)[P_C(\bar{x}_n - e_n) - p]\| \\ &\le \alpha_n\alpha\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|P_C(\bar{x}_n - e_n) - p\|\end{aligned} \tag{2.2}$$

Putting $M = \max\{\|x_0 - p\|, \frac{\|u-p\|}{1-\alpha}\}$, we show that $\|x_n\| \le M$ for all $n \ge 0$. It is easy to see that the result holds for $n = 0$. Assume that the result holds for some $n \ge 0$. That is, $\|x_n - p\| \le M$. Next, we prove that $\|x_{n+1} - p\| \le M$. Indeed, from (2.2), we see that $\|x_{n+1} - p\| \le M$. This shows that the sequence $\{x_n\}$ is bounded. Next, we show that $\limsup_{n\to\infty}\langle u - z, x_{n+1} - z\rangle \le 0$, where $z = \lim_{t\to\infty} J_t f(z)$. From Lemma 1.1, we see that $\lim_{t\to\infty} J_t f(z)$ exists and is the point of $T^{-1}(0)$ nearest to $f(z)$. Since T is maximal monotone, $T_t u \in TJ_t f(z)$ and $T_{\lambda_n}x_n \in TJ_{\lambda_n}x_n$, we see $\limsup_{n\to\infty}\langle f(z) - J_t f(z), J_{\lambda_n}x_n - J_t f(z)\rangle \le 0$. On the other hand, by the nonexpansivity of J_{λ_n}, we obtain $\|J_{\lambda_n}(x_n + e_n) - J_{\lambda_n}x_n\| \le \|(x_n + e_n) - x_n\| = \|e_n\|$. From the assumption $e_n \to 0$ as $n \to \infty$, we arrive at $\limsup_{n\to\infty}\langle f(z) - J_t z, J_{\lambda_n}(x_n + e_n) - J_t f(z)\rangle \le 0$. It follows that

$$\|P_C(\bar{x}_n - e_n) - J_{\lambda_n}(x_n + e_n)\| \le \|(\bar{x}_n - e_n) - J_{\lambda_n}(x_n + e_n)\| \le \|e_n\|.$$

That is, $\lim_{n\to\infty}\|P_C(\bar{x}_n-e_n)-J_{\lambda_n}(x_n+e_n)\|=0$. It follows that

$$\limsup_{n\to\infty}\langle f(z)-J_tf(z),P_C(\bar{x}_n-e_n)-J_tf(z)\rangle\le 0.$$

On the other hand, from the algorithm (2.1), we see that

$$x_{n+1}-P_C(\bar{x}_n-e_n)=\alpha_n[f(x_n)-P_C(\bar{x}_n-e_n)]+\beta_n[x_n-P_C(\bar{x}_n-e_n)].$$

It follows from the condition $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=0$ that $x_{n+1}-P_C(\bar{x}_n-e_n)\to 0\quad$ as $n\to\infty$, It follows that $\limsup_{n\to\infty}\langle f(z)-z,x_{n+1}-z\rangle\le 0$. Notice that

$$\begin{aligned}
\|x_{n+1}-z\|^2&=\langle\alpha_nf(x_n)+\beta_nx_n+(1-\alpha_n-\beta_n)P_C(\bar{x}_n-e_n)-z,x_{n+1}-z\rangle\\
&\le\alpha_n\langle f(x_n)-z,x_{n+1}-z\rangle+\beta_n\langle x_n-z,x_{n+1}-z\rangle\\
&\quad+(1-\alpha_n-\beta_n)\langle P_C(\bar{x}_n-e_n)-z,x_{n+1}-z\rangle\\
&\le\alpha_n\langle f(x_n)-z,x_{n+1}-z\rangle+\beta_n\|x_n-z\|\|x_{n+1}-z\|\\
&\quad+(1-\alpha_n-\beta_n)\|P_C(\bar{x}_n-e_n)-z\|\|x_{n+1}-z\|\\
&\le\alpha_n\langle f(x_n)-z,x_{n+1}-z\rangle+\beta_n\|x_n-z\|\|x_{n+1}-z\|\\
&\quad+(1-\alpha_n-\beta_n)\|x_n-z\|\|x_{n+1}-z\|\\
&=\alpha_n\langle f(x_n)-z,x_{n+1}-z\rangle+(1-\alpha_n)\|x_n-z\|\|x_{n+1}-z\|\\
&\le\alpha_n\langle f(x_n)-z,x_{n+1}-z\rangle+\frac{1-\alpha_n}{2}(\|x_n-z\|^2+\|x_{n+1}-z\|^2).
\end{aligned}$$

This implies that

$$\|x_{n+1}-z\|^2\le(1-\alpha_n)\|x_n-z\|^2+\alpha_n\langle f(z)-z,x_{n+1}-z\rangle.$$

An application of Lemma 1.3, we obtain that $x_n\to z$ as $n\to\infty$. This completes the proof.

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