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CLOSED-FORM SOLUTION FOR GENERALIZED VASICEK DYNAMIC TERM STRUCTURE MODEL WITH TIME-VARYING PARAMETERS AND EXPONENTIAL YIELD CURVES

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Abstract. In this paper I study a generalized Vasicek dynamic term structure model with time-varying parameters, where the short rate *r* is unbounded and the time to maturity for the exponential yield curve model is an exponential function of the short rate. Closed-form solutions are derived for two cases by function analysis technique, with the classical Vasicek equation used as a special case. The methods employed in this paper may have significance in the study of other aspects of finance.

Keywords: generalized Vasicek equation, time-varying parameter, exponential yield curve, closed-form solution.

2010 AMS Subject Classification: 91B28.

1. Introduction

As with many topics in modern finance, the origin of term structure modeling can be traced back to Merton's work. Merton (1974) develops the first so-called short-rate model for the term structure of interest rate. Many refinements have followed, most notably Vasicek (1977), i.e. the Vasicek model. The Vasicek model is a well known mathematical model describing the evolution of interest rates. It specifically describes interest rate movements as driven by a

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single factor - market risk. The Vasicek model is commonly used in the valuation of interest rate derivatives and has also been adapted for credit markets.

Vasicek (1977), which utilizes a Gaussian model, generally assumes the market prices of risk to be constant. The partial differential equation for the classic Vasicek model can be written as

$$P_t + \frac{1}{2}\sigma^2 P_{rr} + aP_r - rP = 0\,, \quad 0 < t < T, \quad -\infty < r < +\infty\,, \tag{1}$$

$$P(T,\ T,\ r) = 1, \qquad -\infty < r < +\infty\,. \tag{2}$$

where P is price, r is the short rate, P_t is the first derivative of price with respect to time, σ^2 is the variance, P_{rr} is the second derivative of price with respect to the short rate, and P_r is first derivative of price with respect to the short rate. The solution to equations (1)-(2) follows the form:

$$P(t,T,r) = \exp[-(T-t)r - \frac{1}{2}a(T-t)^2 + \frac{\sigma^2}{6}(T-t)^3]\,. \tag{3}$$

Despite the utility of this model, there are shortcomings. For example, it is well known that financial markets are affected by numerous factors, many of which relate to time. Babbs and Nowman (1999) present strong evidence in favor of time-varying market prices of risk, thus making constant market prices of risk very difficult to estimate. In order to address this time-related issue, Gaussian Vasicek type models can be modified as suggested by Hull and White (1990) or more recently Dai and Singleton (2002). Further examples of Vasicek models with time-varying parameters include Mamon (2004), which presents three different methods for finding closed-form solutions to bond prices using a Vasicek model. Lemke (2008) investigates an affine macro-finance term structure model for the euro area. Bernaschi, Tacconi and Vergni (2008) present an analysis of the dynamics of the term structure of interest rates based on the study of the time evolution of the parameters of a variation of the Nelson–Siegel model. Dai and Singleton (2000) estimate several affine yield models using the simulation-based efficient method of moments. The authors show that, at least in theory, efficiency can be achieved if the number of moment conditions goes to infinity with the number of data observations. Ait-Sahalia and Kimmel (2010) present a way to estimate affine multifactor term structure models using closed-form likelihood expansions. They develop and implement a technique for closed-form maximum likelihood estimation (MLE) of multifactor affine yield models and derive closed-form approximations to likelihoods for nine Dai and Singleton (2000) affine models.

Current models utilizing an equilibrium term structure for interest rates are often based on a representative agent framework with specific parametric assumptions about the preferences of the representative agent. It is important that when dealing with a Vasicek bond pricing problem with time relationships that the short rate r be unbounded from what essentially remains an open question because the time to maturity is a function of the short rate. Some approaches to this may be seen in Stampfli and Goodman (2001), Christiansen (2005), Lemke (2008), Realdon (2009), Jiang and Yan (2009), Laurini and Moura (2010), Jang and Yoon (2010) , Yu and Zivot(2011), Wang (1996), Honda, Tamaki, and Shioham (2010), Chen, Liu, and Cheng (2010), Mahdavi (2008), and Fouque, Papanicolaou, and Sircar (2000).

The purpose of this paper is to present a theoretical research for generalized Vasicek dynamic term structure models with time-varying parameters, where the short rate r is unbounded and the time to maturity for the exponential yield curve model is an exponential function of the short rate. Closed-form solutions are derived for two cases by function analysis technique, with the classical Vasicek equation used as the special case.

2. Solution for generalized non-homogeneous Vasicek equation

The original Vasicek equation (Equation (1)) utilizes a Taylor expansion technique. Therefore, certain items are left out of the complete equation/series. I next assume these items as a function with time-varying parameters, r and $t\rightarrow$ function $f\,(r,t)$. From this I obtain the generalized non-homogeneous Vasicek equation with time-varying parameters, which can be written as Equation (4).

$$P_t+\frac{1}{2}\sigma^2(t)P_{rr}+a(t)P_r-rP=f(r,t),\quad 0<t<T,\quad -\infty<r<+\infty, \tag{4}$$

$$P(r,T,T)=1,\qquad -\infty<r<+\infty. \tag{5}$$

An emphasis is given in this section to obtain the closed-form exact analytical solution for Equations (4)-(5) and for the special case of $f(r,t)=e^{(t-T)r}g(t)$ by using the variable function analysis technique to derive the closed-form exact analytical solution.

2. 1 Closed-form solution

Consider the characteristic and form of a Vasicek equation with time-varying parameters (Equations (4)-(5)). I assume that the solution of Equations (4)-(5) has the following form:

$$P(r,t,T) = b(t)\exp[A(t)r + B(t)], \tag{6}$$

where $b(t), A(t), B(t)$ are functions which need to be determined. It is from Equation (6) I obtain

$$\begin{aligned}\frac{\partial P}{\partial t} &= b'(t)\exp[A(t)r + B(t)] + b(t)[A'(t)r + B'(t)]\exp[A(t)r + B(t)] \\ &= b'(t)\exp[A(t)r + B(t)] + [A'(t)r + B'(t)]P\end{aligned},$$

$$\frac{\partial P}{\partial r} = b(t)A(t)\exp[A(t)r + B(t)] = A(t)p, \quad \frac{\partial^2 P}{\partial r^2} = A^2(t)P,$$

Taking the above expressions into Equation (4), I obtain

$$\begin{aligned}&b'(t)\exp[A(t)r + B(t)] + [A'(t)r + B'(t)]p + + \frac{1}{2}\sigma^2(t)A^2(t)p \\ &+ a(t)A(t)p - rp = f(r,t),\end{aligned}$$

i.e.,

$$[A'(t)r + B'(t) + \frac{1}{2}\sigma^2(t)A^2(t) + a(t)A(t) - r]p + b'(t)\exp[A(t)r + B(t)] = f(r,t). \tag{7}$$

If $A'(t) = 1$, I obtain $A(t) = t + C$. By choosing $C = -T$, this yields

$$A(t) = t - T. \tag{8}$$

Furthermore, let

$$B'(t) + \frac{1}{2}\sigma^2(t)A^2(t) + a(t)A(t) = 0, \tag{9}$$

i.e.,

$$B'(t) = -a(t)A(t) - \frac{1}{2}\sigma^2(t)A^2(t). \tag{10}$$

Integrating Equation (8) from t to T, I obtain

$$B(t) = B(T) + \int_t^T a(\tau)(\tau - T)d\tau + \frac{1}{2}\int_t^T \sigma^2(\tau)(\tau - T)^2 dt.$$

If $B(T) = 0$, this yields

$$B(t)=\int_t^T a(\tau)(\tau-T)d\tau+\frac{1}{2}\int_t^T \sigma^2(\tau)(\tau-T)^2 d\tau\ , \tag{11}$$

and

$$b'(t)\exp[A(t)r+B(t)]=f(r,t)\ , \tag{12}$$

For $f(r,t)=e^{(t-T)r}g(t)$, the special case, let $b'(t)=e^{-B(t)}g(t)$,

$$b(t)=b(T)-\int_t^T e^{-B(\tau)}g(\tau)d\tau\ .$$

Substituting $b(t)$, $A(t)$ and $B(t)$ into Equation (6), yields $b(T)=1$, i.e.,

$$b(t)=1-\int_t^T e^{-\int_\tau^T[a(s)(s-T)\ +\ \frac{1}{2}\sigma^2(s)(s-T)^2]ds}g(\tau)d\tau\ . \tag{13}$$

It follows from (6) that the solution of Equations (4)-(5) can be written as

$$P(r,t,T)=[1-\int_t^T e^{-\int_\tau^T[a(s)(s-T)\ +\ \frac{1}{2}\sigma^2(s)(s-T)^2]ds}g(\tau)d\tau]$$
$$\cdot\exp[(t-T)r+\int_t^T a(\tau)(\tau-T)d\tau+\frac{1}{2}\int_t^T \sigma^2(\tau)(\tau-T)^2 d\tau].$$

2.2 Specific examples

Utilizing the closed-form solution derived in section 2.1 I give the solution for two specific cases, both of which indicate the advantage of expressing the solution as Equation (13) of this paper.

Example1. Let $a(t)=a$ (constant), $\sigma(t)=\sigma$ (constant) and $g(t)=0$. By doing so I immediately obtain the solution to the classical Vasicek Equations (1)-(2),

$$P(r,t,T)=\exp[-(T-t)r-\frac{1}{2}a(T-t)^2+\frac{\sigma^2}{6}(T-t)^3]\ .$$

with the solution being identical to Equation (3).

Example 2. Let $a(t)=t$, $\sigma(t)=2t^2$ and $g(t)=1$. From this I obtain

$$P(r,t,T)=[1-\int_t^T e^{-\int_\tau^T[s(s-T)\ -\ 2s^4(s-T)^2]ds}d\tau]$$
$$\cdot\exp[(t-T)r+\int_t^T \tau(\tau-T)d\tau+2\int_t^T \tau^4(\tau-T)^2 d\tau],$$

i.e.,

$$P(r,t,T)=[1-\int_t^T e^{-\int_\tau^T[a(s)(s-T)\,+\,\frac{1}{2}\sigma^2(s)(s-T)^2]ds}g(\tau)d\tau]$$
$$\cdot\exp[(t-T)r+\int_t^T a(\tau)(\tau-T)d\tau+\frac{1}{2}\int_t^T\sigma^2(\tau)(\tau-T)^2 d\tau]$$
$$=[1-\frac{1}{420}\exp(t-T)^2(15t^5+5t^4T-2t^3T^2-6t^2T^3-5T(-7+T^4)-7t(-5+T^4)]$$
$$\cdot\exp[(t-T)r+\frac{1}{210}(-70t^3-60t^7+105t^2T+140t^6T-84t^5T^2-35T^3+4T^7)].$$

3. Solution for generalized Vasicek equation with exponential maturity value model

In this section I investigate the generalized Vasicek equation with the exponential maturity value model. The exponential maturity value can be expressed as a function of short rate r . The generalized Vasicek equation can be written as:

$$P_t+\frac{1}{2}\sigma^2(t)P_{rr}+a(t)P_r-rP=0\,,0<t<T,\ -\infty<r<+\infty\,, \tag{14}$$

$$P(r,T,T)=e^{-\lambda r},\qquad -\infty<r<+\infty\,. \tag{15}$$

3.1 Closed-form solution

Consider the characteristic and form of the Vasicek equation with time-varying parameters (4)-(5), I assume that the solution to Equations (14)-(15) has the following form:

$$P(r,t,T)=c(r)b(t)\exp[A(t)r+B(t)]\,, \tag{16}$$

where $c(r),b(t),A(t),B(t)$ are functions which need to be determined. Following from Equation (16), I obtain

$$\begin{aligned}\frac{\partial P}{\partial t}&=c(r)b'(t)\exp[A(t)r+B(t)]+c(r)b(t)[A'(t)r+B'(t)]\exp[A(t)r+B(t)]\\&=c(r)b'(t)\exp[A(t)r+B(t)]+[A'(t)r+B'(t)]P\end{aligned},$$

$$\begin{aligned}\frac{\partial P}{\partial r}&=c'(r)b(t)\exp[A(t)r+B(t)]+c(r)b(t)A(t)\exp[A(t)r+B(t)]\\&=c'(r)b(t)\exp[A(t)r+B(t)]+A(t)p\end{aligned},$$

$$\begin{aligned}\frac{\partial^2 P}{\partial r^2}&=c''(r)b(t)\exp[A(t)r+B(t)]+c'(r)b(t)A(t)\exp[A(t)r+B(t)]\\&\quad+A(t)\{c'(r)b(t)\exp[A(t)r+B(t)]+A(t)P\}\\&=c''(r)b(t)\exp[A(t)r+B(t)]+2c'(r)A(t)b(t)\exp[A(t)r+B(t)]+A^2(t)P.\end{aligned}$$

Taking the expressions above into Equation (14), I obtain

$$c(r)b'(t)\exp[A(t)r+B(t)]+[A'(t)r+B'(t)]p+\frac{1}{2}\sigma^2(t)c''(r)b(t)\exp[A(t)r+B(t)]$$
$$+\sigma^2(t)c'(r)b(t)A(t)\exp[A(t)r+B(t)]+\frac{1}{2}\sigma^2(t)A^2(t)p$$
$$+a(t)c'(r)b(t)\exp[A(t)r+B(t)]+a(t)A(t)p-rp=0,$$

i.e.,

$$[A'(t)r+B'(t)+\frac{1}{2}\sigma^2(t)A^2(t)+a(t)A(t)-r]p$$
$$+\{c(r)b'(t)+[\frac{1}{2}\sigma^2(t)c''(r)+\sigma^2(t)c'(r)A(t)+a(t)c'(r)]b(t)\}\exp[A(t)r+B(t)]=0. \tag{17}$$

If $A'(t)=1$, I obtain $A(t)=t+C$. By choosing $C=-T$, this yields $A(t)=t-T$. Furthermore, let

$$B'(t)+\frac{1}{2}\sigma^2(t)A^2(t)+a(t)A(t)=0, \tag{18}$$

Since $\exp[A(t)r+B(t)]\neq 0$, this yields

$$c(r)b'(t)+[\frac{1}{2}\sigma^2(t)c''(r)+\sigma^2(t)c'(r)A(t)+a(t)c'(r)]b(t)=0. \tag{19}$$

In terms of Equation (8),

$$B'(t)=-a(t)A(t)-\frac{1}{2}\sigma^2(t)A^2(t). \tag{20}$$

Integrating Equation (18) from t to T, I obtain

$$B(t)=B(T)+\int_t^T a(\tau)(\tau-T)d\tau+\frac{1}{2}\int_t^T\sigma^2(\tau)(\tau-T)^2dt.$$

Substituting the expressions of $A(t), B(t)$ into Equation (16), I obtain

$$P(T,T,r)=c(r)b(T)e^{B(T)} \tag{21}$$

Let $c(r)=e^{-\lambda r}$ and $B(T)=0$. In view of condition (5), it yields $b(T)=1$ and

$$B(t)=\int_t^T a(\tau)(\tau-T)d\tau+\frac{1}{2}\int_t^T\sigma^2(\tau)(\tau-T)^2d\tau. \tag{22}$$

Next, I determine the function of $b(t)$. It follows from Equation (19) that

$$\frac{b'(t)}{b(t)}=-\frac{\frac{1}{2}\sigma^2(t)c''(r)+[\sigma^2(t)A(t)+a(t)]c'(r)}{c(r)}.$$

One sees immediately that, for $c(r)=e^{-\lambda r}$,

$$\frac{b'(t)}{b(t)}=-\{\frac{1}{2}\sigma^2(t)\lambda^2-[\sigma^2(t)A(t)+a(t)]\lambda\}.$$

It follows then that

$$\ln|b(t)|=-\int_t^T\{\frac{1}{2}\sigma^2(\tau)\lambda^2-[\sigma^2(\tau)A(\tau)+a(\tau)]\lambda\}d\tau+\ln|c_2|,$$

i.e.,

$$\ln|b(t)|=-\int_t^T\{\frac{1}{2}\sigma^2(\tau)\lambda^2-[\sigma^2(\tau)A(\tau)+a(\tau)]\lambda\}d\tau+\ln|c_2|$$

$$b(t)=c_2e^{-\int_t^T\{\frac{1}{2}\sigma^2(\tau)\lambda^2-[\sigma^2(\tau)A(\tau)+a(\tau)]\lambda\}d\tau}.$$

In view of $c_2=b(T)=1$, this yields

$$b(t)=e^{-\int_t^T\{\frac{1}{2}\sigma^2(\tau)\lambda^2-[\sigma^2(\tau)A(\tau)+a(\tau)]\lambda\}d\tau}.$$

Therefore, I obtain the closed-form solution to a generalized Vasicek dynamic term structure models with time-varying parameters, where the time to maturity is a function of short rate r.

$P(r,t,T)=e^{-\lambda r}e^{-\int_t^T\{\frac{1}{2}\sigma^2(\tau)\lambda^2-[\sigma^2(\tau)(\tau-T)+a(\tau)]\lambda\}d\tau}\exp[(t-T)r+\int_t^T a(\tau)(\tau-T)d\tau+\frac{1}{2}\int_t^T\sigma^2(\tau)(\tau-T)^2d\tau]$, i.e,

$$P(r,t,T)=\exp\left\{(-\lambda+t-T)r-\int_t^T\{\frac{1}{2}[\lambda^2-2(\tau-T)\lambda-(\tau-T)^2]\sigma^2(\tau)+[\lambda-(\tau-T)]a(\tau)\}d\tau\right\}. \tag{23}$$

3.2 Specific examples

The results in section 3 are next used to give the solution for two specific examples, which again indicates the advantage of expressing the solution as Equation (13) of this paper.

Example 1. Let $a(t)=a$ (constant), $\sigma(t)=\sigma$ (constant) and $\lambda=0$. From this I immediately obtain the solution to the classical Vasicek Equations (1)-(2),

$$P(r,t,T)=\exp[-(T-t)r-\frac{1}{2}a(T-t)^2+\frac{\sigma^2}{6}(T-t)^3].$$

with the solution being identical to Equation (3).

Example 2. Let $a(t)=t$, $\sigma(t)=2t^2$ and $\lambda=1$ ($P(r,T)=e^{-r}$), from which I obtain

$$P(r,t,T)=\exp\left\{(-1+t-T)r-\int_t^T \{\,2[1-2(\tau-T)-(\tau-T)^2]\ \tau^4+[1-(\tau-T)]\tau\}d\tau\right\}$$

$$=\exp[(-1+t-T)r-\frac{1}{210}(70t^3+60t^7-140t^6(-1+T)-150t^2(1+T)$$

$$+80t^5(-1-2T+T^2)+T^2(105+35T+84T^3+28T^4-4T^5))].$$

4. Conclusion

In this paper I investigate generalized Vasicek dynamic term structure models with time-varying parameters, where the short rate r is unbounded and the time to maturity for the exponential yield curve model is an exponential function of the short rate. Closed form solutions are derived for two cases by function analysis technique with the classical Vasicek equations as a special case. The mathematical technique employed in this paper may have significance in studying other problems related to financial engineering.

Conflict of Interests

The author declares that there is no conflict of interests.

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