Available online at http://scik.org
Math. Finance Lett. 2014, 2014:3
ISSN 2051-2929

OPTION PRICING IN THE MULTIDIMENSIONAL BLACK-SCHOLES-MERTON MARKET WITH GAUSSIAN HEATH-JARROW-MORTON INTEREST RATES: THE PARSIMONIOUS AND CONSISTENT HULL-WHITE MODELS OF VASICEK AND NELSON-SIEGEL TYPE

WERNER HÜRLIMANN

Wolters Kluwer Financial Services Switzerland AG, Seefeldstr. 69, 8008 Zürich, Switzerland

Abstract: An explicit state-price deflator for the multidimensional Black-Scholes-Merton market with a multiple factor Gaussian bond price dynamics is constructed. It immediately yields an extension of the Margrabe formula in this multiple risk economy. Restricting further the attention to those Gaussian Heath-Jarrow-Morton interest rate models with time-homogeneous sensitivities that share the Markov diffusion property, one is led to consider models of the Hull-White type only. For practical reasons, only consistent families of yield curves are retained. That is, if an initial forward rate curve has a given form, then its future evolution should remain of the same form. Given two simple consistent forms and their most parsimonious parameterizations with two respectively four parameters, as well as their corresponding Hull-White models, we derive an explicit generalized Black-Scholes formula that takes into account the Hull-White term structure.

Keywords: state-price deflator; Black-Scholes formula; Heath-Jarrow-Morton model; Hull-White model; Gaussian interest rates; Vasicek term structure; Nelson-Siegel forward curve

2010 AMS Subject Classification: 60G15; 62P05; 91B24; 91G20; 91G30.

Received March 28, 2014

1. Introduction

The concept of state-price deflator or stochastic discount factor, which has been introduced by Duffie [4], p.23 and 97, is a convenient ingredient of general financial pricing rules. Based on it, the author has introduced a multivariate Black-Scholes deflator and has applied it to option pricing in [11-13]. An extension of the Black-Scholes deflator to a more general version with Vasicek interest rates as additional source of randomness is defined in [14]. It has been used to obtain extensions of the Margrabe and Black-Scholes option pricing formulas and a validation of them in a multiple risk economy with Vasicek interest rates. In particular, the invariance of these formulas against changing market prices of risk, which has been first noticed in [11], is preserved in the extended model.

The present follow-up goes a step further and constructs in Theorem 2.1 a Gaussian state-price deflator for a more general multidimensional Black-Scholes-Merton market with a multiple factor Gaussian bond price dynamics. As shown in Theorem 4.1 Margrabe's formula to exchange one risky asset against the other remains valid in the considered multiple factor Gaussian interest rate environment. The derivation of further generalized versions of the Black-Scholes call option formula under Gaussian interest rates turns out to be more challenging.

The Gaussian Heath-Jarrow-Morton (GHJM) model with multiple factors provides a useful general method to generate Gaussian interest rates. It is considered in Section 3. If one requires that the sensitivities of the GHJM model should be time-homogeneous, and if the short rate should follow a Markov diffusion process, then the class of GHJM models reduces to a Hull-White GHJM interest rate model. In this context, an interesting question of practical relevance concerns the comparison of the initial forward rate curve with its future evolution. For example, given initial parameterizations of the Nelson-Siegel yield curve and a dynamic model of interest rates, e.g. the GHJM Hull-White model, will the future yield curves also be of Nelson-Siegel form? Intuitively, it seems reasonable to use a parameterization, which is consistent with the model dynamics, i.e. with future yields curves of the same form though with possibly different parameter values (e.g. Munk [21], Section 9.7). Some answers to the preceding question, which rely on advanced mathematics, have been given by Björk and Christensen [2] (see also Björk [1], Munk [21], Section 9.7). We consider two simple

forward rate curves that are consistent with the Hull-White model, the one being the simplest consistent form of Nelson-Siegel type. We analyze the conditions under which these forms are consistent and obtain parsimonious parameterizations with two respectively four parameters. The first one is closely related to the term structure model by Vasicek [25] and generates the so-called Hull-White-Vasicek (HWV) model. The second one is consistent with a forward curve of Nelson-Siegel type and is called Hull-White-Nelson-Siegel (HWNS) model. As a main new result of practical interest, Theorem 4.2 states a closed-form formula for the European call option for the multidimensional Black-Scholes-Merton market with HWV or HWNS interest rates.

2. A state-price deflator in a multiple risk economy with Gaussian interest rates

Consider a multiple risk economy with $m \geq 1$ risky assets, whose real-world prices with time horizon T satisfy the stochastic differential equations of Itô type

$$dS_t^{(k)} / S_t^{(k)} = \mu_k(t)dt + \sigma_k dW_t^{(k)}, \quad t \in [0,T], \quad k = 1,...,m, \tag{2.1}$$

where the σ_k's are constant volatilities, the $\mu_k(t)$'s are arbitrary time dependent Gaussian drifts, and the $W_t^{(k)}$'s are correlated standard Wiener processes. The geometric Brownian motions (2.1) constitute a so-called *multidimensional Black-Scholes-Merton market*. The economy contains also an exogenously given money market account, whose value follows the real-world dynamics

$$dM_t / M_t = r_t \cdot dt, \quad t \in [0,T], \tag{2.2}$$

where the *short rate* process r_t follows some well-defined Gaussian process (to be specified in the Sections 3 and 4). Let B_t^T be the price at time t of a zero-coupon bond paying 1 unit of account with certainty at maturity date T. It is assumed that the bond

prices depend upon $n \geq 1$ factors of Gaussian type with volatilities $\sigma_{m+i}(t,T), i = 1,...,n$ such that the following real-world dynamics hold (e.g. Munk [21], Section 4.4.2):

$$dB_t^T / B_t^T = \{r_t - \sum_{i=1}^{n} \lambda_{m+i} \cdot \sigma_{m+i}(t,T)\} \cdot dt - \sum_{i=1}^{n} \sigma_{m+i}(t,T) \cdot dW_t^{(m+i)}, \tag{2.3}$$

where the $W_t^{(m+i)}, i = 1,...,n$ are standard Wiener processes. Moreover, the Wiener processes driving the risky assets and the bond prices are correlated such that $E[dW_t^{(i)} dW_t^{(j)}] = \rho_{ij} dt, 1 \leq i, j \leq m+n$. The constants $\lambda_{m+i} < 0$ are the market prices with respect to the i-th factor of the zero-coupon bond. Note that a constant market price of risk can be justified using market equilibrium theory (e.g. Munk [21], Section 5.4.2). One observes that the terms in the second sum of (2.3) are negative (a positive shock to the short rate implies a negative shock to the zero-coupon bond price and vice versa). Since in equilibrium risky assets have usually an expected rate of return that exceeds the instantaneous risk-free rate, the constants λ_{m+i} must be negative. Consider further the (constant) market prices $\lambda_k > 0$ of the first m risky assets defined by

$$\lambda_k \sigma_k = \mu_k(t) - r(t), \quad k = 1,...,m. \tag{2.4}$$

An application of Itô's Lemma to the system of stochastic differential equations (2.1) and (2.3) (taking into account (2.4)) implies the following representations in terms of the *integrated short rate* process $R_t = \int_0^t r_s ds$. For all $t \in [0,T]$ one has

$$\begin{aligned} S_t^{(k)} &= S_0^{(k)} \cdot \exp\left\{R_t + (\lambda_k \sigma_k - \tfrac{1}{2}\sigma_k^2) \cdot t + \sigma_k \cdot W_t^{(k)}\right\}, \quad k = 1,...,m, \\ B_t^T &= B_0^T \cdot \exp\left\{\begin{array}{l} R_t - \sum_{i=1}^{n} \lambda_{m+i} \cdot \int_0^t \sigma_{m+i}(s,T) ds \\ -\frac{1}{2}\sum_{i=1}^{n} \int_0^t \sigma_{m+i}^2(s,T) ds - \sum_{i=1}^{n} \int_0^t \sigma_{m+i}(s,T) dW_s^{(m+i)} \end{array}\right\}. \end{aligned} \tag{2.5}$$

By assumption, the short rate r_t (given the initial value r_0) is normally distributed. Therefore, the integrated short rate R_t is also normally distributed. It follows that the risky assets in (2.5) are exponential Gaussian processes with lognormal distributions. The state-price deflator in [11,14] generalizes as follows to the context of a multiple risk economy with multiple factor Gaussian interest rates.

Theorem 2.1. (Exponential Gaussian state-price deflator of dimension $m+n$) *Given is a Black-Scholes-Merton market with* $m \geq 1$ *risky assets in a stochastic Gaussian interest rate environment with* $n \geq 1$ *factors. Assume the risky assets and the bond price follow the log-normal real-world prices (2.5), where the correlation matrix* $C = (\rho_{ij}), 1 \leq i, j \leq m+n$ *of the multivariate Wiener process* $W_t = (W_t^{(1)},...,W_t^{(m+n)})$ *is non-singular and positive semi-definite. Then, the exponential Gaussian process given by*

$$\begin{aligned} &D_t^{(m+n)} = \exp(-R_t - \tfrac{1}{2}\beta^T C\beta \cdot t - \beta^T W_t), \\ &\beta = C^{-1}\lambda, \quad \lambda = (\lambda_1,...,\lambda_{m+n})^T, \quad t \in [0,T], \end{aligned} \tag{2.6}$$

is a well-defined state-price deflator.

Proof. According to the general theory of state-price deflators (e.g. Munk [21], Section 4.3), the stochastic process (2.6) defines a deflator provided the following conditions are fulfilled:

*D*1. $D_0^{(m+n)} = 1, \quad D_t^{(m+n)} > 0, \quad \forall t \in [0,T], \quad$ *all states*, $\quad Var[D_t^{(m+n)}] < \infty, \quad \forall t \in [0,T],$

*D*2. $E[D_t^{(m+n)} \exp(R_t)] = 1, \quad \forall t \in [0,T],$

*D*3. $E[D_t^{(m+n)} S_t^{(k)}] = S_0^{(k)}, k = 1,...,m, \quad E[D_t^{(m+n)} B_t^T] = B_0^T, \quad \forall t \in [0,T].$

The first condition D1 is trivially fulfilled. The conditions D2 and D3 mean that the discounted cumulative interest rate process and the discounted risky asset prices are martingales. The validity of D2 follows from the fact that $\ln\{D_t^{(m+n)} \exp(R_t)\} = -\frac{1}{2}\beta^T C\beta \cdot t - \beta^T W_t$ is normally distributed with mean $-\frac{1}{2}\beta^T C\beta \cdot t$ and variance $\beta^T C\beta \cdot t$. To show the first part of D3 let $m_k(t)$ and $v_k^2(t)$ be the mean and variance of the normally distributed random variable $X_t^{(k)} = \ln\{D_t^{(m+n)} S_t^{(k)}\} = \ln S_0^{(k)} - \frac{1}{2}\beta^T C\beta \cdot t - \beta^T W_t + (\lambda_k \sigma_k - \frac{1}{2}\sigma_k^2) \cdot t + \sigma_k \cdot W_t^{(k)}$. One has

$$m_k(t) = \ln S_0^{(k)} - \tfrac{1}{2}\beta^T C\beta \cdot t + (\lambda_k \sigma_k - \tfrac{1}{2}\sigma_k^2)\cdot t, \quad k = 1,\ldots,m,$$

$$v_k^2(t) = \beta^T C\beta \cdot t + \sigma_k^2 t - 2\cdot \sum_j \rho_{kj}\beta_j \sigma_k t = \beta^T C\beta \cdot t + (\sigma_k^2 - 2\lambda_k \sigma_k)\cdot t,$$

where the last equality follows from the fact that $C\beta = \lambda$. It follows that

$$E[D_t^{(m+n)} S_t^{(k)}] = E[\exp(X_t^{(k)})] = \exp\left\{m_k(t) + \tfrac{1}{2}v_k^2(t)\right\} = S_0^{(k)},\ k = 1,\ldots,m.$$

Similarly, let $m_B(t)$ and $v_B^2(t)$ be the mean and variance of the normal random variable

$$Y_t = \ln\left\{D_t^{(m+n)} B_t^T\right\} = \ln B_0^T - \tfrac{1}{2}\beta^T C\beta \cdot t - \beta^T W_t - \sum_{i=1}^n \lambda_{m+i}\cdot \int_0^t \sigma_{m+i}(s,T)ds$$
$$- \tfrac{1}{2}\sum_{i=1}^n \int_0^t \sigma_{m+i}^2(s,T)ds - \sum_{i=1}^n \int_0^t \sigma_{m+i}(s,T)dW_s^{(m+i)}.$$

From the rules of stochastic calculus (e.g. Munk [21], Chapter 3), one gets

$$m_B(t) = \ln B_0^T - \tfrac{1}{2}\beta^T C\beta \cdot t - \sum_{i=1}^n \lambda_{m+i}\cdot \int_0^t \sigma_{m+i}(s,T)ds - \tfrac{1}{2}\sum_{i=1}^n \int_0^t \sigma_{m+i}^2(s,T)ds,$$

$$v_B^2(t) = \beta^T C\beta \cdot t + \sum_{i=1}^n \int_0^t \sigma_{m+i}^2(s,T)ds + 2\cdot \sum_{i,j} \rho_{j,m+i}\beta_j \cdot \int_0^t \sigma_{m+i}(s,T)ds$$
$$= \beta^T C\beta \cdot t + \sum_{i=1}^n \int_0^t \sigma_{m+i}^2(s,T)ds + 2\cdot \sum_{i=1}^n \lambda_{m+i}\cdot \int_0^t \sigma_{m+i}(s,T)ds,$$

where again the equation $C\beta = \lambda$ has been used. It follows that

$$E[D_t^{(m+1)} B_t^T] = E[\exp(Y_t)] = \exp\left\{m_B(t) + \tfrac{1}{2}v_B^2(t)\right\} = B_0^T.$$

This shows D3 and completes the proof. ◊

3. The parsimonious and consistent Hull-White models of Vasicek and Nelson-Siegel type

In the *Gaussian Heath-Jarrow-Morton* (GHJM) model with $n \geq 1$ factors the

T-maturity *forward rates* risk-neutral Q-dynamics is described by the Itô process (e.g. [21], Section 10.5)

$$f_t^T = f_0^T + \int_0^t \hat{\alpha}(u,T)du + \sum_{i=1}^n \int_0^t \beta_i(u,T)dW_u^{Q(m+i)}, \tag{3.1}$$

where the forward rate sensitivities $\beta_i(t,T)$ at time t depend only on the maturity date T, and the following drift restriction holds (e.g. Munk [21], Theorem 10.2, equation (10.5))

$$\hat{\alpha}(u,T) = \sum_{i=1}^n \beta_i(u,T) \cdot \int_u^T \beta_i(u,s)ds. \tag{3.2}$$

Since the $\beta_i(u,T)$'s depend only on time, the stochastic integrals in (3.1) are normally distributed (e.g. Munk [21], Theorem 3.3). Therefore, the future forward rates (3.1) are normally distributed under the Q-measure. The short rate process $r_t = f_t^t$ is then also normally distributed under the Q-measure and given by

$$r_t = f_0^t + \int_0^t \hat{\alpha}(u,t)du + \sum_{i=1}^n \int_0^t \beta_i(u,t)dW_u^{Q(m+i)}, \quad t \geq 0. \tag{3.3}$$

It is natural to ask about reasonable properties a GHJM model should satisfy. First, one may argue that the sensitivities should be *time-homogeneous*, that is do not depend on calendar time and therefore of the form $\beta_i(t,T) = \beta_i(T-t)$ for some time to maturity dependent functions $\beta_i(\tau), \tau = T-t$ (consult the discussions in Munk [21], Chapter 9, and Chapter 10, p.296). Another attractive property is the restriction to Markov diffusion processes of the short rate with dynamics of the type

$$dr_t = \mu(t, r_t)dt + \sum_{i=1}^{n} \sigma_i(t, r_t) dW_u^{Q(m+i)}, \quad t \geq 0. \tag{3.4}$$

The class of GHJM models with both time-homogeneous and Markov diffusion short rates reduces to a one-factor GHJM model with a time-homogenous sensitivity function of the type

$$\beta(\tau) = \sigma \cdot e^{-\kappa\tau}, \quad \sigma > 0, \kappa \geq 0, \quad \tau = T - t. \tag{3.5}$$

According to Munk [21], p.296, this important result is due to Hull and White [10]. If $\kappa = 0$ in (3.5) this leads to the Ho and Lee [7] extension of the model of Merton [15]. If $\kappa > 0$ the model identifies with the Hull and White [9] extension of the model by Vasicek [25] studied in Munk [21], Section 10.4.2. While the Hull-White specification (3.5) is the most tractable one, there exist a plenty of other more complex but still tractable GHJM models (e.g. Munk [21], Sections 10.6.3, 10.7, 10.8, Exercise 10.1). An attractive non-Markovian GHJM model is the time-homogeneous humped volatility model introduced independently by Mercurio and Moraleda [18] and Ritchken and Chuang [24] with specification

$$\beta(\tau) = (\sigma + \gamma \cdot \tau) \cdot e^{-\kappa\tau}, \quad \sigma, \gamma, \kappa \geq 0, \gamma > \kappa \text{ if } \gamma > 0, \quad \tau = T - t. \tag{3.6}$$

If $\gamma = 0$ one recovers the Hull-White model (3.5). If $\sigma = \kappa = 0$ this model coincides with the specification by Nielsen and Sandmann [23]. The humped volatility model is discussed further in Mercurio and Moraleda [19] and Moraleda and Vorst [20]. The consistency and calibration of this model is discussed in Falco et al. [5]. For simplicity, and unless otherwise stated, we consider throughout the Hull-White specification (3.5).

To begin, let us briefly discuss the following general question, which is of practical relevance for model calibration (e.g. Munk [21], Section 9.7, Falco et al. [5]). It concerns the comparison of the initial forward curve f_0^T with its future evolution f_t^T in (3.1). For example, consider the initial parameterization by Nelson and Siegel [22] of the form

$$f_0^T = c_0 + c_1 e^{-\kappa T} + c_2 T e^{-\kappa T}, \tag{3.7}$$

and suppose a GHJM model, say the Hull-White model, is given. Then, one can ask whether the future yield curve $f_t^T, t \in [0,T]$, will also be of the Nelson-Siegel form (although possibly with different parameter values), that is such that

$$f_t^T = C_0 + C_1 e^{-\kappa(T-t)} + C_2 (T-t) e^{-\kappa(T-t)} + \sigma \cdot \int_0^t e^{-\kappa(T-u)} dW_u^Q . \tag{3.8}$$

For an affirmative answer, one says that the initial parameterization is consistent with the given GHJM model. The original Nelson-Siegel form (3.7) is not consistent with the Hull-White model (e.g. Björk and Christensen [2], Björk [1], Munk [21], Section 9.7). Even more, it is not consistent with any non-trivial diffusion model, a result due to Filipovic [6]. Fortunately, there exist initial parameterizations, which are consistent with the GHJM Hull-White model. Two consistent forward rate curves are (e.g. Munk [21], end of Section 9.7)

$$f_0^T = c_1 e^{-\kappa T} + c_2 e^{-2\kappa T}, \tag{3.9}$$

$$f_0^T = c_0 + c_1 e^{-\kappa T} + c_2 T e^{-\kappa T} + c_3 e^{-2\kappa T} . \tag{3.10}$$

The form (3.10) is the simplest consistent form of Nelson-Siegel type extending (3.7). Before proceeding with the separate analysis of (3.9) and (3.10), we need explicit formulas for (3.1) and (3.2) for the one-factor Hull-White specification (3.5). The drift restriction (3.2) yields

$$\hat{\alpha}(u,T) = \sigma^2 \cdot e^{-\kappa(T-u)} \cdot \int_u^T e^{-\kappa(s-u)} ds = (\sigma^2 / \kappa) \cdot e^{-\kappa(T-u)} (1 - e^{-\kappa(T-u)}) . \tag{3.11}$$

Inserted into (3.1) one obtains the future forward rates in the Q-measure as

$$\begin{aligned} f_t^T &= f_0^T + \tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2 \cdot \{e^{-2\kappa T} - 2e^{-\kappa T} + 2e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}\} + \sigma \cdot e^{-\kappa T} X_t^Q, \\ X_t^Q &= \int_0^t e^{\kappa u} dW_u^Q. \end{aligned} \quad . \qquad (3.12)$$

3.1. The consistent Hull-White-Vasicek interest rate model

To analyze under which conditions the form (3.9) will be consistent with the Hull-White model, insert it into (3.12) to see that

$$\begin{aligned} f_t^T &= \{c_1 - \left(\tfrac{\sigma}{\kappa}\right)^2\} \cdot e^{-\kappa T} + \{c_2 + \tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2\} \cdot e^{-2\kappa T} \\ &+ \left(\tfrac{\sigma}{\kappa}\right)^2 \cdot e^{-\kappa(T-t)} - \tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2 \cdot e^{-2\kappa(T-t)} + \sigma \cdot e^{-\kappa T} X_t^Q. \end{aligned} \qquad (3.13)$$

Now, this is of the consistent form

$$f_t^T = c_1 e^{-\kappa(T-t)} + c_2 e^{-2\kappa(T-t)} + \sigma \cdot e^{-\kappa T} X_t^Q \qquad (3.14)$$

if, and only if, one sets

$$c_1 = \left(\tfrac{\sigma}{\kappa}\right)^2, \quad c_2 = -\tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2. \qquad (3.15)$$

Clearly, the short rate in the Q-measure of this model is given by

$$r_t = f_t^t = \theta + \sigma \cdot e^{-\kappa T} X_t^Q, \quad r_0 = \theta = \tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2, \qquad (3.16)$$

and satisfies the stochastic differential equation (SDE)

$$dr_t = \kappa(\theta - r_t)dt + \sigma \cdot dW_t^Q, \qquad (3.17)$$

which characterizes the term structure model by Vasicek [25]. However, in contrast to its original version, the parameter identification $\theta = \tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2$ gives a meaning to this constant and reduces the number of required parameters to two. Henceforth, it is natural to call this simplest parsimonious and consistent interest rate model the *Hull-White-Vasicek* (HWV) model.

3.2. The consistent Hull-White-Nelson-Siegel interest rate model

Next, we analyze the conditions under which the form (3.10) is consistent with the Hull-White model. For this, insert it into (3.12) to see that

$$f_t^T = c_0 + \{c_1 - \left(\tfrac{\sigma}{\kappa}\right)^2\} \cdot e^{-\kappa T} + c_2 \cdot Te^{-\kappa T} + \{c_3 + \tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2\} \cdot e^{-2\kappa T} + \left(\tfrac{\sigma}{\kappa}\right)^2 \cdot e^{-\kappa(T-t)} - \tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2 \cdot e^{-2\kappa(T-t)} + \sigma \cdot e^{-\kappa T} X_t^Q. \tag{3.18}$$

A parsimonious model is obtained by setting

$$c_1 = \left(\tfrac{\sigma}{\kappa}\right)^2, \quad c_3 = -\tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2. \tag{3.19}$$

Assuming this, the equation (3.18) rewrites as

$$f_t^T = c_0 + \left(\tfrac{\sigma}{\kappa}\right)^2 \cdot e^{-\kappa(T-t)} + c_2 \cdot Te^{-\kappa T} - \tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2 \cdot e^{-2\kappa(T-t)} + \sigma \cdot e^{-\kappa T} X_t^Q. \tag{3.20}$$

With the identity $c_2 \cdot Te^{-\kappa T} = c_2 e^{-\kappa t}(T-t)e^{-\kappa(T-t)} + c_2 te^{-\kappa t}e^{-\kappa(T-t)}$ one gets the form

$$f_t^T = C_0 + C_1 e^{-\kappa(T-t)} + C_2(T-t)e^{-\kappa(T-t)} + C_3 e^{-2\kappa(T-t)}, \tag{3.21}$$

which is consistent with (3.10) for the time t dependent parameters

$$C_0 = c_0, \quad C_1 = \left(\tfrac{\sigma}{\kappa}\right)^2 + c_2 te^{-\kappa t}, \quad C_2 = c_2 e^{-\kappa t}, \quad C_3 = -\tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2. \tag{3.22}$$

Now, change notation and set $\alpha = c_0$, $\beta = c_2$, $\theta = \tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2$. Then, the short rate reads

$$r_t = f_t^t = \alpha + \theta + \beta te^{-\kappa t} + \sigma \cdot e^{-\kappa T} X_t^Q, \quad r_0 = \alpha + \theta. \tag{3.23}$$

Through application of Itô's Lemma one sees that the short rate satisfies the SDE

$$dr_t = \kappa(\theta(t) - r_t)dt + \sigma \cdot dW_t^Q, \quad \theta(t) = \alpha + \theta + \beta\kappa^{-1}e^{-\kappa t}. \tag{3.24}$$

Clearly, the special case $\alpha = \beta = 0$ coincides with the HWV model of Section 3.1. In

general, the parameterization $(\alpha, \beta, \kappa, \sigma)$ of the initial forward curve

$$f_0^T = \alpha + \beta T e^{-\kappa T} + 2\theta e^{-\kappa T} - \theta e^{-2\kappa T}, \quad \theta = \tfrac{1}{2}\left(\tfrac{\sigma}{\kappa}\right)^2, \tag{3.25}$$

is as parsimonious as the original Nelson-Siegel forward curve, but it has the advantage to be consistent with the Hull-White model. It is therefore natural to call this simplest consistent extension of the original Nelson-Siegel model the *Hull-White-Nelson-Siegel* (HWNS) model.

4. Option pricing in the Gaussian interest rate environment

Closed-form Margrabe and European option pricing formulas on risky assets in the multidimensional Black-Scholes-Merton market with Gaussian interest rates are obtained. The new results are closely related to Theorems 3.1 and 4.1 in [14]. We begin with the Margrabe formula, which remains valid for all multiple factor Gaussian interest rate processes.

Theorem 4.1. (Margrabe's formula in a multiple factor Gaussian interest rate environment) *Under the assumptions of Theorem 2.1, the market value at initial time of a European exchange option on the risky assets with real-world prices* $S_t^{(k)}, S_t^{(\ell)}, k \neq \ell \in \{1,...,m\}$ *and strike time* T *is given by the formula*

$$\begin{aligned} &E[D_T^{(m+n)}(S_T^{(k)} - S_T^{(\ell)})_+] \\ &= S_0^{(k)}\Phi\left(\frac{\ln\left\{S_0^{(k)}/S_0^{(\ell)}\right\} + \frac{1}{2}\upsilon^2 T}{\upsilon\sqrt{T}}\right) - S_0^{(\ell)}\Phi\left(\frac{\ln\left\{S_0^{(k)}/S_0^{(\ell)}\right\} - \frac{1}{2}\upsilon^2 T}{\upsilon\sqrt{T}}\right), \\ &\upsilon^2 = \sigma_k^2 + \sigma_\ell^2 - 2\rho_{k\ell}\sigma_k\sigma_\ell, \quad k \neq \ell \in \{1,...,m\}. \end{aligned} \tag{4.1}$$

Proof. Using the generalized exponential Gaussian state-price deflator of Theorem 2.1, this is identical to the derivation of Theorem 3.1 in [14]. ◊

The comments of Remarks 3.1 in [14] remain true in the present context. We derive now generalized versions of the European call option formula by Black and Scholes [3] (see also

Merton [16]), to a multiple risk economy with one-factor HWV and HWNS interest rate models. The elementary derivation is done along the line of [14], Section 4. Starting point is the short rate dynamics (3.24) of the HWNS model under the Q-measure, which includes as special case $\alpha = \beta = 0$ the one-factor HWV model. With the usual change of measure the dynamics under the real-world P-measure turns out to be

$$dr_t = \kappa\big(\theta(t) + \lambda_{m+1}\sigma\kappa^{-1} - r_t\big)\cdot dt + \sigma \cdot dW_t^{(m+1)}. \tag{4.2}$$

The following formulas enter into the generalized Black-Scholes formula (4.4) below.

Lemma 4.1. *Assume the real-world short rate process follows the HWNS stochastic differential equation (4.2) and let* $R_T = \int_0^T r_s ds$ *be the associated integrated short rate process at maturity date* T*, and set* $B(T) = \kappa^{-1}(1 - e^{-\kappa T})$*. Then, the following identities hold true:*

$$\begin{aligned}
&\text{F1.} \quad E[R_T] = E^Q[R_T] + \lambda_{m+1}\cdot Cov[R_T, W_T^{(m+1)}],\\
&\qquad\quad Cov[R_T, W_T^{(m+1)}] = \sigma\kappa^{-1}\cdot\{T - B(T)\},\\
&\text{F2.} \quad E^Q[R_T] = (\theta + \alpha - \beta\kappa^{-1}e^{-\kappa T})\cdot T + \beta\kappa^{-1}\cdot B(T),\\
&\text{F3.} \quad \mathrm{Var}[R_T] = (\sigma\kappa^{-1})^2\cdot\{T - B(T) - \tfrac{1}{2}\kappa\cdot B(T)^2\}.
\end{aligned} \tag{4.3}$$

Proof. For simplicity in notation we omit indices and set $\lambda = \lambda_{m+1}$, $W_t = W_t^{(m+1)}$. Integrating both sides of (4.2) over the interval $[0,T]$ yields the identity

$$r_T - r_0 = \kappa\left(\int_0^T \theta(s)ds + \lambda\sigma\kappa^{-1}T\right) - \kappa\cdot\int_0^T r_s ds + \sigma\kappa^{-1}\cdot\int_0^T dW_s\,.$$

Using the definition of the integrated process and rearranging one gets the representation

$$R_T = \big(\theta + \alpha + \lambda\sigma\kappa^{-1}\big)\cdot T + \beta\kappa^{-1}B(T) + \kappa^{-1}\cdot\{r_0 - r_T + \sigma\cdot W_T\}.$$

On the other hand (4.2) is the SDE of an Ornstein-Uhlenbeck process, whose solution can be written as

$$r_t = (r_0 - h(0)) \cdot e^{-\kappa t} + h(t) + \sigma \cdot e^{-\kappa t} X_t, \quad X_t == \int_0^t e^{\kappa u} dW_u ,$$

where the function $h(t)$ solves the ordinary differential equation

$$\kappa^{-1} h'(t) + h(t) = \theta(t) + \lambda\sigma\kappa^{-1}, \quad \theta(t) = \alpha + \theta + \beta\kappa^{-1} e^{-\kappa t}.$$

Indeed, setting $g(t) = (r_0 - h(0)) \cdot e^{-\kappa t} + h(t)$, one obtains from Itô's Lemma that $r_t = g(t) + \sigma \cdot e^{-\kappa t} X_t$ solves the SDE

$$dr_t = \kappa\left(\kappa^{-1} g'(t) + g(t) - r_t\right) \cdot dt + \sigma \cdot dW_t = \kappa\left(\theta(t) + \lambda\sigma\kappa^{-1} - r_t\right) \cdot dt + \sigma \cdot dW_t ,$$

where the second equality follows from the fact that

$$\begin{aligned} &\kappa^{-1} g'(t) + g(t) = -(r_0 - h(0)) \cdot e^{-\kappa t} + \kappa^{-1} h'(t) + (r_0 - h(0)) \cdot e^{-\kappa t} + h(t) \\ &= \kappa^{-1} h'(t) + h(t) = \theta(t) + \lambda\sigma\kappa^{-1} \end{aligned}$$

With $h(t) = \alpha + \theta + \beta t e^{-\kappa t} + \lambda\sigma\kappa^{-1}$ the solution of (4.2) reads

$$r_t = r_0 \cdot e^{-\kappa t} + (\theta + \alpha + \lambda\sigma\kappa^{-1})(1 - e^{-\kappa t}) + \beta t e^{-\kappa t} + \sigma \cdot e^{-\kappa t} X_t .$$

Inserting into the equation for R_T and taking into account that $r_0 = \alpha + \theta$ by (3.23) one obtains the (short rate independent) representation

$$R_T = \left(\theta + \alpha + \lambda\sigma\kappa^{-1} - \beta\kappa^{-1} e^{-\kappa T}\right) \cdot T + (\beta\kappa^{-1} - \lambda\sigma\kappa^{-1}) \cdot B(T) + \sigma\kappa^{-1} \cdot \left\{W_T - e^{-kT} X_T\right\}$$

Formulas for the first terms of the identities F1 and F2 follow at once, namely

$$\begin{aligned} &E[R_T] = (\theta + \alpha - \beta\kappa^{-1} e^{-\kappa T}) \cdot T + \beta\kappa^{-1} \cdot B(T) + \lambda\sigma\kappa^{-1} \cdot \{T - B(T)\}, \\ &E^Q[R_T] = (\theta + \alpha - \beta\kappa^{-1} e^{-\kappa T}) \cdot T + \beta\kappa^{-1} \cdot B(T). \end{aligned}$$

Clearly, the risk-neutral integrated mean in F2 follows by setting $\lambda = 0$ in the real-world integrated mean. To calculate the variance of the integrated process one notes that

$$\begin{aligned}&\mathrm{Var}[R_T] = (\sigma\kappa^{-1})^2 \cdot \mathrm{Var}[W_T - e^{-kT} X_T] \\ &= (\sigma\kappa^{-1})^2 \cdot \{T - 2e^{-\kappa T} \cdot Cov[W_T, X_T] + e^{-2\kappa T} \cdot Var[X_T]\}.\end{aligned}$$

But, one has

$$Var[X_T] = \int_0^T e^{2\kappa s} ds = (1/2\kappa) \cdot (e^{2\kappa T} - 1), \quad Cov[W_T, X_T] = \int_0^T e^{\kappa s} ds = (1/\kappa) \cdot (e^{\kappa T} - 1).$$

Insert into the preceding relation and rearrange to get formula F3. The remaining covariance is calculated as follows (use the explicit representation of R_T):

$$\begin{aligned}&Cov[R_T, W_T] = \mathrm{E}[R_T W_T] = \sigma\kappa^{-1} \cdot \mathrm{E}[W_T^2 - e^{-\kappa T} W_T X_T] \\ &= \sigma\kappa^{-1} \cdot \{T - e^{-\kappa T} Cov[W_T, X_T]\} = \sigma\kappa^{-1} \cdot \{T - B(T)\}.\end{aligned}$$

The proof of Lemma 4.1 is complete.

The following generalized version of Theorem 4.1 in [14] follows.

Theorem 4.2. (European option for the multidimensional Black-Scholes-Merton market with HWV or HWNS interest rates) *Under the assumptions of Theorem 2.1, the market value at initial time of a European call option on the risky asset with real-world price* $S_t^{(k)}$, $k \in \{1,...,m\}$, *strike time* T *and strike price* K *is given by the closed-form formula*

$$E[D_T^{(m+1)} (S_T^{(k)} - K)_+] = S_0^{(k)} \cdot \Phi(d_1^{(k)}) - K \cdot \exp\{-r_f(T)\} \cdot \Phi(d_2^{(k)}), \qquad (4.4)$$

with

$$\begin{aligned}&d_1^{(k)} = \frac{\ln\{S_0^{(k)} / K\} + r_f(T) + \frac{1}{2}\upsilon_k^2(T)}{\upsilon_k(T)}, \quad d_2^{(k)} = d_1^{(k)} - \upsilon_k(T), \\ &r_f(T) = E^Q[R_T] - \tfrac{1}{2}\mathrm{Var}[R_T], \qquad (4.5) \\ &\upsilon_k^2(T) = \sigma_k^2 \cdot T + Var[R_T] + 2\rho_{1,m+1}\sigma_{\mathrm{k}} \cdot Cov[R_T, W_T^{(m+1)}],\end{aligned}$$

and these quantities are determined by Lemma 4.1.

Proof. Using the generalized exponential Gaussian state-price deflator of Theorem 2.1, this is identical to the derivation of Theorem 3.1 in [14]. ◊

To conclude the exposé we would like to challenge interested readers to derive with the state-price deflator technique a call option pricing formula for a risky asset under the non-Markovian one-factor GHJM interest rate model with time-homogeneous humped volatility (3.6).

Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES

[1] T. Björk, On the geometry of interest rate models. In: R.A. Carmona, E. Cinlar, I. Ekeland, E. Jouini, J.A. Scheinkman, N. Touzi (Ed.), Paris-Princeton Lectures on Mathematical Finance 2003, Lecture Notes in Mathematics 1847, Springer, NY, 2003.

[2] T. Björk, B.J. Christensen, Interest rate dynamics and consistent forward rate curves, Mathematical Finance 9(4), (1994), 323-348.

[3] F. Black, M. Scholes, The pricing of options and corporate liabilities, Journal of Political Economy 81, (1973), 637-59, reprinted in [8].

[4] D. Duffie, Dynamic Asset Pricing Theory, Princeton University Press, New Jersey, 1992.

[5] A. Falco, L.I. Navarro, J. Nave, On the calibration of a Gaussian Heath-Jarrow-Morton model using consistent forward rate curves, Quantitative Finance 11(4), (2011), 495-504.

[6] D. Filipovic, A note on the Nelson-Siegel family. Mathematical Finance 9(4), (1999), 349-359.

[7] T.S.E. Ho, S.-B. Lee, Term structure movements and pricing interest rate contingent Claims, Journal of Finance 41(5), (1986), 1011-1029.

[8] L. Hugston (Editor), Options: Classic Approaches to Pricing and Modelling, Risk Books, 1999.

[9] J. Hull, A. White, Pricing interest rate derivatives securities, Review of Financial Studies 3(4), (1990), 573-592.

[10] J. Hull, A. White, One-factor interest rate models and the valuation of interest-rate derivative Securities, Journal of Financial and Quantitative Analysis 28(2), (1993), 235-254.

[11] W. Hürlimann, An extension of the Black-Scholes and Margrabe formulas to a multiple risk economy, Applied Mathematics 2(4), (2011), 137-142.

[12] W. Hürlimann, Analytical pricing of an insurance embedded option: alternative formulas and Gaussian approximation, Journal of Informatics and Mathematical Science 3(2), (2011), 87-105.

[13] W. Hürlimann, The algebra of option pricing: theory and application, International Journal of Algebra and Statistics 1(2), (2012), 68-88.

[14] W. Hürlimann, Option pricing in the multidimensional Black-Scholes market with Vasicek interest rates, Mathematical Finance Letters 2(1), (2013), 1-18.

[15] R.C. Merton, A dynamic general equilibrium model of the asset market and its application to the pricing of the capital structure of the firm, WP 497-70, Sloan School of Management, MIT, (1970), reprinted in [16].

[16] R.C. Merton, Theory of Rational Option Pricing, Bell Journal of Economics and Management Science 4(1), (1973), 141–183, reprinted in [17].

[17] R.C. Merton, Continuous-Time Finance, Basil Blackwell, 1992.

[18] F. Mercurio, J. Moraleda, An analytically tractable interest rate model with humped volatility, European Journal of Operational Research 120, (2000), 205-214.

[19] F. Mercurio, J. Moraleda, A family of humped volatility models, The European Journal of Finance 7, (2001), 93-116.

[20] J. Moraleda, T. Vorst, Pricing American interest rate claims with humped volatility models, Journal of Banking and Finance 21(8), (1997), 1131-1157.

[21] C. Munk, Fixed Income Modelling, Oxford University Press, New York, 2011.

[22] C.R. Nelson, A.F. Siegel, Parsimonious modelling of yield curves, Journal of Business 60(4), (1987), 473-489.

[23] J.A. Nielsen, K. Sandmann, Equity-linked life insurance – a model with stochastic interest rates, Insurance: Mathematics and Economics 16(3), (1995), 225-253.

[24] P. Ritchken, I. Chuang, Interest rate option pricing with volatility humps, Review of Derivative Research 3, (1999), 237-262.

[25] O.A. Vasicek, An equilibrium characterization of the term structure, Journal of Financial Economics 5, (1977), 177-188.