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# OPTION PRICING UNDER TWO-STATE MARKOV CHAIN MARKET MODEL

PETAR RADKOV

Faculty of Mathematics and Informatics, Sofia University ”St. Kl. Ohridski”, Sofia 1407, Bulgaria



**Abstract.** This paper analyses a two-state Markov chain model, which is a discrete-time model of a financial market. The uncertainty in a financial market is presented as the changes of the risky asset are modulated by a discrete-time, two-state, Markov chain. It examines two versions of our Markov chain market model: first, where the model has a recombinant tree, and second, with a non-recombinant tree. Risk-neutral probability measure in the Markov chain market model was also discussed and defined. Considering the European call option in the case of recombinant tree, which is the simplest departure from independency of underlying asset from the classical option price model, the risk neutral probability measure is the same as in the Cox-Ross-Rubinstein model, and consequently the price of option. In the case of non-recombinant tree a method for valuation of option in the Markov chain model using calibration to the market option price is presented. The suggested two-state Markov chain market model has the bull and bear features of the underlying asset price fluctuations and it gives better results with the evaluation of option price of companies from DJIA.

**Keywords:** Markov chain market; option pricing; recombinant and non-recombinant tree; correlated Bernoulli trials.

**2010 AMS Subject Classification:** 60J05, 60J10, 91B24.

## 1. Introduction

One of the basic model of financial market in discrete-time is the binomial model; see, e.g. Cox et al. (1979) [7]. The classical option pricing formula in discrete-time model in Cox,

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Ross and Rubinstein is based on a the main assumption that ups and downs of stock prices are independent, having sequence of independent Bernoulli trials. In this paper, for simplicity, it is considered a financial market model for a single stock and risk-free asset. It is used the two-state Markov chain which govern the realization of price changes of risky asset. The paper discusses risk neutral probability and option pricing under recombinant and non-recombinant tree.

Several option pricing models have been proposed that allow for serial dependence of the underlying asset's returns. Discrete-time Markov chain models provide an important class of asset price models. They have been considered by authors such as Pliska (1997) [19], Norberg (2003) [16] and van der Hoek and Elliott (2010) [24]. Omey and Gulck (2006) [17] have also generalized the classical binomial approach of the model of Black and Scholes to a Markov binomial approach.

Another approach is by using Markov and semi-Markov processes, where dependence on the past in the underlying asset model is explicitly account for. Such strategy has been used by D'Amico et al. (2009) [8]. Other related models include Song et al. (2010) [15] for a multivariate Markov chain asset price models and Valakevicius (2009) [23] for a continuous-time Markov chain asset price models.

One of the key motives for considering Markov chain asset price models is that discrete-time Markov chain can provide a reasonable approximations to continuous-time diffusion processes. The valuation of some complex options may be more simple in a discrete-time Markov chain asset price model.

This paper is organized as follows. Section 2 presents two-state Markov chain market model with recombinant tree. Here it is given the definition of two-state Markov chain. Also define a two-state Markov chain market in recombinant case and the price dynamics of stock. It discusses the issue of defining and estimating a risk neutral probability measure in the Markov chain market in recombinant framework. Section 3 defines a two-state Markov chain market and the price dynamics of stock but in the framework of a new model, which has a non-recombinant tree. Here again it is discussed the choice of a risk neutral probability measure in the Markov chain market and and its risk neutral measure transition matrix. In this it is explained method for pricing a European style call option using a non-recombinant tree and calibration to the market option price. Also its gives the algorithm for finding non- recombinant tree and its probability density function. Section 4 it is compared the prices of European call options calculated under our two state Markov chain market model with non-recombinant tree and Black-Scholes model. The final section summarises the paper.

## 2. A two-state Markov chain model with recombinant tree

This section presents a discrete-time Markov chain market model in the framework of recombinant tree, where the randomness of the price process of share is modeled by a discrete-time, two state, time-homogeneous Markov chain. Similar models were discussed in some recent work as Valakevicius (2009) [23], Song at al. (2010) [15] and van der Hoek and Elliott (2010, 2011) [24, 11].

A model with recombinant tree is defined any model where if the risky asset moves up and then down, the price will be the same as if it had moved down and then up.

### 2.1. Two-state Markov Chain

Let consider a complete probability space $(\Omega, \mathscr{F}, P)$, where $P$ is a real world probability measure. It is denoted with $\mathscr{T}$ time parameter set as $\mathscr{T} := \{0,1,2,\ldots,T\}$, where $T$ is a finite positive integer.

To describe uncertainty in Markov chain market, it is considered a discrete-time, two-state, time-homogeneous Markov chain $\{X_n\}_{n\in\mathscr{T}}$ . Following the convention in Elliott et al (1995) [10] it is identified the state space of chain $\{X_n\}_{n\in\mathscr{T}}$ with canonical state space given by the set of standard unit vectors in $\mathbb{R}^2$ :

$$\mathscr{E} = \{\ell_1, \ell_2\},$$

where $\ell_i$ for each $i = 0,1$ are unit vectors in $\mathbb{R}^2$ with unity as the $i+1$th element and zeros elsewhere. In our case has two unit vectors:

$$\ell_1 = (1,0)' \text{ and } \ell_2 = (0,1)',$$

where with $x'$ is denoted the transpose vector of $x$.

Supposing that $X_0$ is given, or its distribution known, the probability law of our Markov chain can be defined as:

**Definition 2.1** To describe the probability law of the chain, first is defined the initial state probability as:

$$\pi_i := P(X_0 = \ell_i),$$

where $i = 0,1$. And, second, is defined the following transition probability:

$$p_{ji} = P(X_{n+1} = \ell_j | X_n = \ell_i), \tag{1}$$

where $i, j = 1,2$, and transition matrix is

$$P = \begin{pmatrix} p_{00} & p_{01} \\ & \\ p_{10} & p_{11} \end{pmatrix}. \tag{2}$$

From this definition it follows that $p_{ji}$ satisfies:

$$p_{ji} \geq 0 \ (j,i = 0,1) \quad \text{and} \quad \sum_{j=0}^{1} p_{ji} = 1 \ (i = 0,1).$$

**Lemma 2.1** *If we consider that the state equation is*

$$X_n = PX_{n-1} + m_n, \tag{3}$$

*then the $m_n$ is martingale increment w.r.t.* $(P, \mathscr{F}_n)$

**Proof.**

$$
\begin{aligned}
E\left[m_n|\mathscr{F}_n\right] &= E\left[X_n - PX_{n-1}|\mathscr{F}_n\right] \\
&= E\left[X_n - PX_{n-1}|X_n\right] \\
&= PX_{n-1} - PX_{n-1} = 0.
\end{aligned}
$$

The basic idea is that it is assumed that two states exist at discrete time $n \in \mathscr{T}$. It is denoted the states as $\{X_n\}_{n\in\mathscr{T}}$ and write $F = \sigma(X_0, X_1, \ldots, X_T)$. By definition the state space of $X_n$ is $\{\ell_0 \text{ and } \ell_1\}$ where $\ell_0 = (1,0)'$, which is called the state "failure" and $\ell_1 = (0,1)'$, which is "success". The Markov chain $X_n$ is equivalent to a sequence of binary random variables $(\nu_n,\ n = 1,2,\ldots)$, defined in Omey et al. (2008) [18], Minkova and Omey (2012) [14] and Minkova and Radkov (2010, 2011) [20, 13], where for a given $\pi \in (0,1)$, the states 1 and 0 appear with initial probabilities $P(\nu_n = 1) = \pi$ and $P(\nu_n = 0) = 1-\pi$. Suppose that the correlation coefficient is $\rho = Corr(\nu_n, \nu_{n-1}),\ n = 2,3,\ldots$. Then, the sequence $\nu_n$ forms two-state Markov chain with transition probabilities

$$P(\nu_{n+1} = 1|\nu_n = 1) = 1-(1-\pi)(1-\rho);$$

$$P(\nu_{n+1} = 0|\nu_n = 0) = 1-\pi(1-\rho),$$

where $\rho \in (\max\{-1, -\frac{1-\pi}{\pi}, -\frac{\pi}{1-\pi}\}, 1)$ and $n = 1,2,\ldots$.

The state 1 of the define sequence $(\nu_n,\ n = 1,2,\ldots)$ could be seen as "success" and the state 0 as a "failure".

## 2.2. Model with recombinant tree

This section considers a discrete-time model of a financial market with the set of dates $\mathscr{T} := \{0,1,2,\ldots,T\}$ with a risky asset $S$, referred to as a stock. It is supposed that the risk-free rate is $r \in (0,1)$.

The price process is:

$$S_n = (1+\rho_n)S_{n-1},\ \ n = 1,2,\ldots,N,\ \ S_0 = S, \tag{4}$$

where $\rho_n = \log(S_n/S_{n-1})$ are the risky asset return. Now, let define a risky asset return process $\{\rho_n\}_{n\in\mathscr{T}}$ by assuming that it can only take value of a finite set values $\mathscr{R} = \{b,a\} \in (-\infty,+\infty)$ and $a<b$. Then in our model risky asset return $\{\rho_n\}_{n\in\mathscr{T}}$ is governed by the Markov chain $\{X_n\}$. For convenience it is defined also the vector $\mathbf{m}$ to be $m = \mathbf{1}+\mathbf{p}$ as $\mathbf{1} = (1;1)$, where 1 is ones vector. Then $\mathbf{m}$ is defined as $\mathbf{m} := (d,u)'$, where $u = \mathbf{1}+b$ and $d = \mathbf{1}+a$. The vector $\mathbf{m}$ represent different factors by which the price can change at any time step depending previous step. So the finite set of possible returns and return factors could be given by vectors:

$$\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix} d \\ u \end{pmatrix} \tag{5}$$

The basic idea is to be modulated return of risky asset with Markov chain $\{X_n\}$. Definition 2.1 see also Omey and Gulck [17] and Minkova and Radkov [20, 13], where is considered the case where ups and downs of return of risky asset is not independent and follow our Markov chain.

So, if the return increase from step $n-1$ to step $n$, then the price of risky asset can change from $S_n$ to $uS_n$ with probability $p_{11}$ according to the definition of the Markov chain (Definition 2.1) and from $S_n$ to $dS_n$ with probability $p_{01}$. If the return decrease from step $n-1$ to $n$, then the price of risky asset can be $dS_n$ with probability $p_{00}$ and $uS_n$ with probability $p_{10}$. At the same time the process of return at the starting point $t=0$ follows the initial probabilities so the risky asset can rise to $uS_n$ with probability $\pi$ and fall to $dS_n$ with probability $1-\pi$.

Then the risky asset return process $\{\rho_n\}$ is governed by the Markov chain $X_n$ by the following way:

**Definition 2.2.** Risky asset return process $\{\rho_n\}$ is governed by the Markov chain $X_n$ by:

$$\rho_n = \langle \mathbf{p}, X_n \rangle, \tag{6}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product.

So the risky asset return process $\rho_n$ is again a discrete-time, two-state Markov chain. Then using (4) and (6) the price process of risky asset could be defined.

**Definition 2.3** Risky asset price process $\{S_n\}_{n\in\mathscr{T}}$ is given by equation

$$S_n = S\prod_{k=1}^{n}\langle \mathbf{m}, X_n\rangle, \tag{7}$$

where $S$ is the risky asset price at $n = 0$.

**Proof.** Risky asset price process generally is given by

$$S_n = S\prod_{k=1}^{n}(1+\rho_k)$$

and using (6) have

$$\begin{aligned} S_n &= S\prod_{k=1}^{n}(1+\rho_k) = S\prod_{k=1}^{n}(1+\langle \mathbf{p}, X_n\rangle) \\ &= S\prod_{k=1}^{n}(\langle \mathbf{1}+\mathbf{p}, X_n\rangle) = S\prod_{k=1}^{n}\langle \mathbf{m}, X_n\rangle. \end{aligned}$$

## 2.2. Risk neutral probability measure

This section presents a measure change for the Markov chain and finding the new probability measure which is risk neutral. Following Elliott et al. (1995) [10]

### 2.2.1. A measure change

Suppose we have a matrix $\mathbf{C} := (c_{ij})_{i,j=1,2}$, which is 2x2 matrix with real-value entities such as:

(1) $0 \le c_{ij} \le 1$,
(2) $\sum_{j=1}^{2} c_{jk} = 1$ for $k = 1 \text{and} k = 2$.

This matrix $\mathbf{C}$ can be a candidate of transition probability matrix of the Markov chain. Define, for each $s = 1, 2, \ldots, T$,

$$\lambda_s := \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{c_{ji}}{p_{ji}}\langle X_s, \ell_j\rangle\langle X_{s-1}, \ell_i\rangle,$$

where it is assumed that $p_{ji} > 0$ for each $i, j = 0, 1$, so that $\lambda_s$ is well defined.

Let consider an $\{\mathscr{F}\}_t$-adapted process $\{\Lambda_n\}_{n\in\mathscr{T}}$ defined by:

$$\Lambda_n := \prod_{k=1}^{2} \lambda_k; \;\; \Lambda_0 = 1.$$

The new probability measure $Q$ on $\mathscr{F}_n$ is defined by putting the restriction of the Radon-Nikodym derivative $dQ/dP$ to $\sigma(\mathscr{F}_n)$ equal to $\Lambda_n$ ($Q \sim P$)

$$\frac{dQ}{dP}\bigg| \mathscr{F}_n := \Lambda_n,$$

for all $n \in \mathscr{T}$.

The existence of $Q$ follows from Kolmogorov's Extension Theorem.

**Lemma 2.2** $\{\Lambda_n\}$ *is an* $(\{\mathscr{F}_n\}, P)-$*martingale.*

The next proposition gives the dynamics of the chain $\{X_n\}$, under the new measure $Q$. This result can be found in Elliott et al. (1995) [10].

**Proposition 2.1** *Under the measure* $Q$, $\{X_n\}_{n\in\mathscr{T}}$ *is a Markov chain with transition probability matrix* $\boldsymbol{C}$.

## 2.2.2. Risk neutral transition matrix

To determine the price on option in the Markov chain market, it is needed to determine a transition matrix under a risk neutral probability measure $Q$ of the form introduced in Proposition 2.1. The fundamental theorem of asset pricing by Harrison and Kres (1979) [12] and Harrison and Pliska (1981, 1983) [19] state that the absence of an arbitrage opportunities is equivalent to the existence of an equivalent martingale measure under which discounted price process are martingales.

This martingale condition is equivalent to having that if $Q$ is an equivalent martingale measure, then

$$S_n = E^Q\left[e^{-r}S_{n+1}\middle|\mathscr{F}_n\right], \;\; n = 1, 2, \ldots, T, \tag{8}$$

where $E^Q$ is an expectation under risk neutral martingale measure $Q$.

The following proposition gives the martingale condition in the Markov chain market.

**Proposition 2.2** *Suppose Q is an equivalent measure of the form introduced in Proposition 2.1 so that under Q, X is a Markov chain with a transition matrix* $\boldsymbol{C}$. *Then Q is an equivalent martingale measure if*

$$e^{-r}\langle \boldsymbol{m}, \boldsymbol{C}\ell_k\rangle - 1 = 0 \tag{9}$$

*for all* $k = 1, 2$.

**Proof.** Using (7) and the Markov property,

$$S_n = E^Q\left[e^{-r}S_{n+1} \middle| \mathscr{F}_n\right].$$

That is,

$$\begin{aligned}
S\prod_{k=1}^{n}\langle \mathbf{m}, X_n\rangle &= E^Q\left[e^{-r}S\prod_{k=1}^{n+1}\langle \mathbf{m}, X_k\rangle |\mathscr{F}_n\right] \\
&= E^Q\left[e^{-r}S\prod_{k=1}^{n}\langle \mathbf{m}, X_k\rangle \langle \mathbf{m}, X_{n+1}\rangle |\mathscr{F}_n\right] \\
&= E^Q\left[e^{-r}S\prod_{k=1}^{n}\langle \mathbf{m}, X_k\rangle \langle \mathbf{m}, X_{n+1}|\mathscr{F}_n\rangle\right] \\
&= e^{-r}S\prod_{k=1}^{n}\langle \mathbf{m}, X_k\rangle \left\langle \mathbf{m}, E^Q\left[X_{n+1}|\mathscr{F}_n\right]\right\rangle \\
&= e^{-r}S\prod_{k=1}^{n}\langle \mathbf{m}, X_k\rangle \left\langle \mathbf{m}, E^Q\left[X_{n+1}|X_n\right]\right\rangle \\
&= e^{-r}S\prod_{k=1}^{n}\langle \mathbf{m}, X_k\rangle \langle \mathbf{m}, \mathbf{C}X_n\rangle.
\end{aligned}$$

Then

$$\left[ S \prod_{k=1}^{n} \langle \mathbf{m}, X_k \rangle \right] \left[ e^{-r} \langle \mathbf{m}, \mathbf{C} X_n \rangle - 1 \right] = 0,$$

$$e^{-r} \langle \mathbf{m}, \mathbf{C} X_n \rangle - 1 = 0$$

or

$$e^{-r} \langle \mathbf{m}, \mathbf{C} \ell_k \rangle - 1 = 0$$

for all $k = 1, 2$.

**Proposition 2.3** *In the case of two-state Markov chain the "risk neutral" transition probability matrix* $\mathbf{C}$ *can be determined uniquely as*

$$C = \begin{pmatrix} \alpha & 1-\beta \\ \\ 1-\alpha & \beta \end{pmatrix}, \tag{10}$$

*where*

$$\alpha = \frac{e^r - d}{u - d}, \qquad \beta = \frac{u - e^r}{u - d}. \tag{11}$$

This result is standard and easy to obtain using the Proposition 2.2, so the result is stated without providing the proof.

**Theorem 2.1** *In two-state Markov chain with recombinant tree the Equivalent martingale measure (EMM) is the same as in the classical binomial option price model and the European style option is given by the famous Cox-Ross-Rubinstein formula.*

**Proof.** This result is easy to obtain using the Proposition 2.3. Calculating risk neutral probability matrix $\mathbf{C}$ we could see that the column vectors are the same.

The Proposition 2.2 is true also for the general case of N-state Markov chain not only in our Markov chain market model with two-state Markov chain. In the case of N-state, the transition probability matrix $\mathbf{C}$ cannot be determined uniquely, see e.g. Elliott et al. (2011) [11].

# 3. A two-state Markov chain model with non recombinant tree

This section presents a discrete-time Markov chain market model, where the randomness of the price process of share is modeled again by a discrete-time,two state, time-homogeneous Markov chain, but this time in the framework of non-recombinant tree. Such models were discussed in Bhat (2012) [1] and Charalambous (2008) [5].

A model with non-recombinant tree is defined as any model where if the risky asset moves up and then down, the price will not be the same as if it had moved down and then up.

## 3.1. Two-state Markov Chain

Let consider the same discrete-time, two-state, time-homogeneous Markov chain $\{X_n\}_{n\in\mathscr{T}}$ as in the previous section. Complete probability space is $(\Omega,\mathscr{F},P)$, where $P$ is a real world probability measure and $\mathscr{T}$ is time parameter set as $\mathscr{T} := \{0,1,2,\dots,T\}$, where $T$ is a finite positive integer. The probability law of the chain is defined in Definition 2.1.

## 3.2. Model with non-recombinant tree

This section considers a discrete-time model of a financial market with the set of dates $\mathscr{T} := \{0,1,2,\dots,T\}$ and with a risky asset $S$. Suppose that the risk-free rate is $r\in(0,1)$.

The price process is:

$$S_n = (1+\rho_n)S_{n-1},\;\; n=1,2,\dots,N,\;\; S_0 = S, \tag{12}$$

where $\rho_n = \log(S_n/S_{n-1})$ are the risky asset return. A risky asset return process $\{\rho_n\}_{n\in\mathscr{T}}$ may be defined by assuming that it can only take a value from a finite set values $\mathbb{R} = \{a,\ b,\ f,\ g\} \in (-\infty,+\infty)$ and $b\ >\ a$ and $g\ >\ f$ which is given in matrix $\mathbf{p}$. For $n=0$ return process $\{\rho_n\}$ can only take a value from a finite set values $\mathbb{R}_1 = \{u',\ d'\} \in (-\infty,+\infty)$ and $d'\ <\ u'$ which is given also in vector $\mathbf{p}'$

For convenience it is also defined the matrix $\mathbf{m}$ and vector $\mathbf{m}'$ here $\mathbf{m} = \mathbf{1} + \mathbf{p}$ where $\mathbf{1}$ is 2x2 matrix of ones, where every element is equal to one and $\mathbf{m}' = \mathbf{1} + \mathbf{p}'$, where $\mathbf{1}$ is ones vector. Then the finite set of possible elements of $\mathbf{p}$ and $\mathbf{m}$ is given by

$$\mathbf{p} = \begin{pmatrix} a & f \\ \\ b & g \end{pmatrix} \quad \text{and} \quad \mathbf{m} = \begin{pmatrix} x & w \\ \\ y & v \end{pmatrix}.$$

The idea is to modulate uncertainty in risky asset returns not only with respect to probabilities of the next step depending from the previous one by Markov chain $\{X_n\}$, but also to introduce dependence in the size of return changes. Above, $a$, $b$, $f$ and $g$ denote the different percentage price changes by which the risky asset at every time step is allowed to change. With symbols $x$, $y$, $w$ and $v$ it is quoted the different factors by which the price can change at any time step depending on the previous step. The factors are equal to the sum of one and percentage price changes. Similar Markov chain model could be found in Bhat and Kummar (2012) [1]

So if the return increase from step $n-1$ to step $n$, then the price of risky asset can change from $S_n$ to $wS_n$ with probability $p_{11}$ according to the definition of the Markov chain (Definition 2.1) and from $S_n$ to $vS_n$ with probability $p_{01}$. If the return decrease from step $n-1$ to $n$, then the price of risky asset can be $xS_n$ with probability $p_{00}$ and $yS_n$ with probability $p_{10}$. At the same time the process of return at the starting point $t = 0$ follows the initial probabilities, so that the risky asset can rise from $S_n$ to $uS_n$ with probability $\pi$ and fall to $dS_n$ with probability $1-\pi$.

Then the risky asset return process $\{\rho_n\}$ is governed by the Markov chain $X_n$ in the following way.

**Definition 3.1** Risky asset return process $\{\rho_n\}$ is governed by the Markov chain $X_n$ by:

$$\rho_1 = \langle \mathbf{p}', X_1 \rangle \qquad \rho_n = \langle \mathbf{p}X_{n-1}, X_n \rangle \quad n = 2, 3, \cdots, T. \tag{13}$$

Then using (12) and (13) the price process of a risky asset could be defined as follows.

**Definition 3.2** A risky asset price process $\{S_n\}_{n\in\mathscr{T}}$ is given by equation

$$S_n = S \langle \mathbf{m}', X_1 \rangle \prod_{k=2}^{n} \langle \mathbf{m}X_{n-1}, X_n \rangle, \tag{14}$$

where $S$ is the risky asset price at $n=0$.

**Proof.** Risky asset price process generally is given by

$$S_n = S\prod_{k=1}^{n}(1+\rho_k)$$

and using (13) have

$$\begin{aligned} S_n &= S\prod_{k=1}^{n}(1+\rho_k) = S\left(1+\left\langle \mathbf{p}', X_1\right\rangle\right)\prod_{k=2}^{n}(1+\langle \mathbf{p}X_{k-1}, X_k\rangle) \\ &= S\left(\left\langle 1+\mathbf{p}', X_1\right\rangle\right)\prod_{k=2}^{n}(\langle 1+\mathbf{p}X_{k-1}, X_k\rangle) = S\left(\left\langle \mathbf{m}', X_1\right\rangle\right)\prod_{k=2}^{n}(\langle \mathbf{m}X_{k-1}, X_k\rangle). \end{aligned}$$

Regarding our Markov chain financial market non-recombinant tree Lemma 2.2 and Proposition 2.2 is true, because is changed only the size of moves of the sequence $\{\rho_n\}$, but the probability law which governs this sequence again is the two-state Markov chain defined by $X_n$.

## 3.3. Risk neutral transition matrix

Using the martingale condition (7), which guarantee that $Q$ is an equivalent martingale measure (EMM) in our Markov chain market, it is derived the following proposition:

**Proposition 3.1** *Suppose $Q$ is an equivalent measure of the form introduce in Proposition 2.1 so that under $Q$, $X$ is a Markov chain with transition matrix $\boldsymbol{C}$. Then $Q$ is a equivalent martingale measure if*

$$e^{-r}\langle \boldsymbol{m}\ell_k, \boldsymbol{C}\ell_k\rangle - 1 = 0 \tag{15}$$

*for all $k=1,2.$ and $n=2,3,\cdots,T$ and*

$$e^{-r}\left\langle \boldsymbol{m}', \boldsymbol{D}\right\rangle - 1 = 0 \tag{16}$$

*for $n=1$.*

**Proof.** Using (14) and the Markov property (8) have:

$$\begin{aligned} S\left\langle \mathbf{m}', X_0\right\rangle \prod_{k=2}^{n} \left\langle \mathbf{m}X_{k-1}, X_k\right\rangle &= E^Q\left[e^{-r} S\left\langle \mathbf{m}', X_0\right\rangle \prod_{k=2}^{n+1} \left\langle \mathbf{m}X_{k-1}, X_k\right\rangle | \mathscr{F}_n\right] \\ &= E^Q\left[e^{-r} S\left\langle \mathbf{m}', X_0\right\rangle \prod_{k=2}^{n} \left\langle \mathbf{m}X_{k-1}, X_k\right\rangle \left\langle \mathbf{m}X_n, X_{n+1}\right\rangle | \mathscr{F}_n\right] \\ &= e^{-r} S\left\langle \mathbf{m}', X_0\right\rangle \prod_{k=2}^{n} \left\langle \mathbf{m}X_{k-1}, X_k\right\rangle \left\langle E^Q\left[\mathbf{m}X_n | \mathscr{F}_n\right], E^Q\left[X_{n+1} | \mathscr{F}_n\right]\right\rangle \\ &= e^{-r} S\left\langle \mathbf{m}', X_0\right\rangle \prod_{k=2}^{n} \left\langle \mathbf{m}X_{k-1}, X_k\right\rangle \left\langle \mathbf{m}X_n, E^Q\left[X_{n+1} | X_n\right]\right\rangle \\ &= e^{-r} S\left\langle \mathbf{m}', X_0\right\rangle \prod_{k=2}^{n} \left\langle \mathbf{m}X_{k-1}, X_k\right\rangle \left\langle \mathbf{m}X_n, CX_n\right\rangle. \end{aligned}$$

Then

$$\left[S\left\langle \mathbf{m}', X_0\right\rangle \prod_{k=2}^{n} \left\langle \mathbf{m}X_{k-1}, X_k\right\rangle\right] \left[e^{-r}\left\langle \mathbf{m}X_n, \mathbf{C}X_n\right\rangle - 1\right] = 0,$$

$$e^{-r}\left\langle \mathbf{m}X_n, \mathbf{C}X_n\right\rangle - 1 = 0$$

or $e^{-r}\left\langle \mathbf{m}\ell_k, \mathbf{C}\ell_k\right\rangle - 1 = 0$ for all $k = 1, 2.$ and $n = 2, 3, \cdots, T$

**Proposition 3.2** *In the case of two-state Markov chain the "risk neutral" transition probability matrix* $\boldsymbol{C}$ *in* $n = 2, 3, \cdots, T$ *and initial "risk neutral" probability matrix* $\boldsymbol{D}$ *in* $n = 1$ *for our Markov chain market non-recombinant tree can be determined uniquely as follows:*

$$C = \begin{pmatrix} \alpha & 1-\beta \\ 1-\alpha & \beta \end{pmatrix} \quad D = \begin{pmatrix} q \\ 1-q \end{pmatrix}, \tag{17}$$

*where*

$$q = \frac{e^r - d}{u - d} \tag{18}$$

*and*

$$\alpha = \frac{y - e^r}{y - x}, \qquad \beta = \frac{e^r - w}{v - w}. \tag{19}$$

This result is standard and easy to obtain using the Proposition 2.3 and (14), so the result is stated without providing the proof.

The Proposition 2.2 is true also for the general case of N-state Markov chain not only in our Markov chain market model with two-state Markov chain. In the case of N-state the transition probability matrix **C** cannot be determined uniquely, see Elliott et al (2011) [11].

## 3.2. Pricing European style call option in two-state Markov chain with non-recombinant tree

Suppose the risk-free rate is $r$, strike price is $K$ and $\tau \in \mathscr{T}$ is the time to maturity. Theoretical price of European call is given by

$$c(K,\mathbf{C},\tau) = e^{-t\tau}E^{Q}\left(S_T - K\right)^{+} = e^{-t\tau}E^{Q}\left[(S_\tau - K)^{+}\right].$$

Following Bhat and Kummar (2012) [1] the European call option price can be given by

$$c(K,\mathbf{C},\tau) = e^{-t\tau}\sum_{k=1}^{J}\left(S_k - K\right)^{+}P^{Q}(S_k), \tag{20}$$

where $J$ is the number of all allowed prices at time to maturity and $S_k$ is equal to any of these possible prices as $k = 1,\ldots,J$.

In the pricing process three decisions need to be made. First, choosing how to find the possible states of the risky asset. The second decision is concerned with the evaluation of the risk neutral probability measure stock to be at any of these allowed states. The third, one is how to estimate the parameters of the model with respect to transition matrix **C**.

The pricing procedure is the following:

First, using the proposed numerical algorithm by us, we produce a list of all allowed states. The procedure is to find the possible combinations of $w$, $v$, $y$, and $x$ during the path of risky asset. Details about the procedure can be found in the Appendix. Then, knowing that the price dynamics followed (14) it can be calculated all allowed states for $n$ steps by

$$S_k = Se^{\mathbf{s}\cdot\mathbf{m}}, \tag{21}$$

where $\mathbf{s} = (\ln u,\ \ln d,\ \ln w,\ \ln v,\ \ln y,\ \ln x)$ and $\mathbf{m} = (m_0,\ m_1,\ m_{11},\ m_{10},\ m_{01},\ m_{00})$ as $m_0$ and $m_1$ denote the possible outcome for the first step which can be 1 or 0 meaning "success" and "failure", $m_{11},\ m_{10},\ m_{01}$ and $m_{00}$ denote the numbers of subsequences of the form "success" - "success", "success" - "failure", "failure" - "success" and "failure" - "failure". Then for $n$ steps we have the following: $m_{00}+m_{10}+m_{01}+m_{00} = n-1$. Hence enumerating all possible vectors is equivalent to enumerate all possible outcomes of $S_k$.

Second, calculate the probability to be at any allowed state under the risk neutral probability measure. Using the Definition 2.1 of our Markov chain the probability can be presented as

$$P(S_k = Se^{\mathbf{s}\cdot\mathbf{m}}) = \mathbf{q}^{\mathbf{m}}, \tag{22}$$

where $\mathbf{q} = (q,\ 1-q,\ \alpha,\ 1-\alpha,\ \beta,\ 1-\beta)$ and $\mathbf{q} = \prod_{i=1}^{6} \mathbf{q}_i^{\mathbf{m}_i}$. Having all of this the entire p.m.f. of $?S_k$ is fully determined.

Third, estimate the parameters of the model ($u,\ d,\ w,\ v,\ y$ and $x$). Then using (18) and (19) can be found "risk-neutral" probability measure **C** and **D**. Bhat and Kummar propose $d = 1/u,\ v = 1/w,\ x = 1/y,\ u = e^{\sigma\sqrt{\tau}},\ w = e^{\sigma_+\sqrt{\tau}}$ and $y = e^{\sigma_-\sqrt{\tau}}$, where $\sigma$ is a standard deviation in an annual basis of risky asset returns, $\sigma_+$ and $\sigma_-$ are a standard deviation in an annual basis of positive returns and strictly negative returns.

The paper proposes another approach for estimating unknown parameters $u,\ d,\ v,\ w,\ x$ and $y$. Calibrating some of these parameters to the market option prices data. The basic idea of calibration is to use market option prices to select a set of parameters in the model dynamics so as to minimize the errors between the theoretical price implied by the model and the observed market option prices. Let, we assume that $d = 1/u,\ w = 1/v$ and $x = 1/y$, taking $u = e^{\sigma\sqrt{\tau}}$. Then, we should determine only two parameters $v$ and $y$ which by applying (18) and (19) guarantee that the transition probability matrix is the "risk neutral". This method of finding $v$ and $x$ determines uniquely the transition matrix **C** in the risk-neutral probability measure in our Markov chain market model. The idea is to determine $v$ and $y$, so that the sum of squared deviation of market option price from the theoretical ones is minimized. In practice, such a calibration of option price is usually done using price data of simple option, such as a standard European call or put option.

Then the risk-neutral probability matrix $\mathbf{C} := (c_{ji})_{j,i=1,2}$ and unknown parameters ($v$ and $x$) can be determined using the following conditions: $0 \le c_{ij} \le 1$ for all $i, j = 1, 2$; $\sum_{j=1}^{2} c_{jk} = 1$ for $k = 1$ and $k = 2$; $e^{-r}\langle \mathbf{m}, \mathbf{C}\ell_k \rangle - 1 = 0$ for all $k = 1$ and $k = 2$ and

$$\sum_{j=1}^{J}\sum_{i=1}^{M}\left(c(K_j, \mathbf{C}, \tau_i) - c_{market}(K_j, \tau_i)\right)^2$$

is minimized for given market option price for different strike prices $K_1, K_2, K_3, \dots, K_J$ and time to maturities $\tau_1, \tau_2, \tau_3, \dots, \tau_M$, where $(v,x)$ define by (18) and (19) as $v > 1 > w$ and $y > 1 > x$ determine uniquely the transition matrix in risk-neutral probability measure $\mathbf{C}$ .

Implying the last condition it is used the minimization square deviation price calibration to select a risk-neutral measure. A similar approach can be found in Cont and Tankov (2006) [22] and Elliott at el.(2011) [11], where the least-square calibration was used to find a risk-neutral measure to price options under an exponential Levy process. According to Carr and Cousot (2011) [4] discrete-time, finite-state, Markov chain asset pricing models are among the very few models which are arbitrage-free and can be calibrated to the finite number of observed market option price. This approach is used also in works of Carr and Madan (2005) [3], Buehler (2006) [2], Cousot (2007) [6], and Davis and Hobson(2007) [9].

## 4. Numerical example and results

In this section, it is compared the prices of European call options calculated under our two state Markov chain market model with non-recombinant tree and Black-Scholes model. The model is mark-to-marketed by minimizing the sum-of-squares distance between the theoretical option price and market option prices for European call options. The estimated parameters are then used to calculate the value for European call options.

It is estimate parameters of the model ($u$, $v$ and $y$) for the all 30 companies in index "DJIA". The data is downloaded from yahoo.finance. The market value of options with maturity 1-month, 3-month and 6-month is for 6 June 2014 and the stock price is for the period 1 December 2008 - 6 June 2012.

Figure 1 and 2 show the the price of a European call option for ”JP Morgan Chase Co. Common S” (JPM) and ”Microsoft” (MSFT) determined by two-state Markov chain model with non-recombinant tree, Black-Scholes model and the market option price. The price of option is given as a function of strike level.

To measure the performance of suggested option prices model and Black-Scholes model are used average absolute error (AAE), average percentage error (APE), and root-mean error (RMSE). There are defined by Schoutens (2003) [21] as follows:

$$AAE = \sum_{j=1}^{J} \left| \frac{M_j - m_j}{J} \right|, \; APE = \frac{1}{mean\ option\ price} \sum_{j=1}^{J} \left| \frac{M_j - m_j}{J} \right|$$

$$\text{and } RMSE = \sqrt{\sum_{j=1}^{J} \frac{\left(M_j - m_j\right)^2}{J}},$$

where $M_j$ are the market option price, $b_j$ are option price of the two-state Markov chain model with non-recombinant tree or Black-Scholes option price model for a different strike levels $j = 1, \ldots J$.

All parameters of the model ($u$, $v$ and $y$) and defined errors for the following companies ”American Express Company Common”(APX), ”Cisco Systems, Inc.”(CSCO), ”General Electric Company Common”(GE), ”International Business Machines”(IBM), ”Intel Corporation”(INTC), ’JP Morgan Chase Co. Common S” (JPM), ”Coca-Cola Company (The) Common”(KO), ”3M Company Common Stock”(MMM), ”Merck Company, Inc. Common St”(MRK), ”Microsoft” (MSFT), ”Pfizer, Inc. Common Stock”(PFE), ”Procter Gamble Company”(PG), ”ATT Inc.”(T), ”The Travelers Companies, Inc. C”(TRV), ”Wal-Mart Stores, Inc. Common St”(WMT), ”Exxon Mobil Corporation Common”(XOM) are shown. The errors are determined by two-state Markov chain model with non-recombinant tree, Black-Scholes model and the market option price.

The values of these errors and estimated parameters for both models are presented in Tables 1 through 4.

Results from errors show that, for the 30 stocks, two-state Markov chain model with non-recombinant tree gets closer option value to the market price of option than the price calculated by the Black-Schole model. The values of total summarized results are shown in Table 5.

## 6. Conclusion

The paper considers a discrete-time, two-state Markov chain market model with two traded securities: risky asset and risk-free asset. The uncertainty in the financial market is introduced as the daily changes of the risky asset are modulated by a discrete-time, two-state, Markov chain.

The issue of selecting risk-neutral measure in the Markov chain financial market model was also discussed. The method for valuation of a European style call option in our Markov chain market model is examined and presented.

Thus, by suitable specifications of parameters (u, v and y) the two-state Markov chain financial market model can be constructed to reflect a possible feature of a real market, having bullish or bearish trends. It could create forgetful markets and markets with long memory, markets with different sorts of dependencies between assets returns. So, the two-state Markov chain market model has the bull and bear features of underlying asset price fluctuations, and it gives better results with the evaluation of option price of companies from DJIA.

Further, the two-state Markov chain financial market model could be used for evaluating different path dependent options, especially the American option, as the two-state Markov chain model with non-recombinant tree is calibrated to the current option market price.

# Appendix

Here, it is presented in short the proposed method for finding all possible outcomes of a risky asset for $n$ steps. With $m_0$ we define the outcome of the first step, which can be 1 or 0 meaning "success" and "failure". Let $m_{11},\ m_{10},\ m_{01}$ and $m_{00}$ denote the number of subsequences of the form "success" - "success", "success" - "failure", "failure" - "success" and "failure" - "failure". Having total $n$ steps we have the following:

$$m_{11} + m_{10} + m_{01} + m_{00} = n - 1 \tag{22}$$

Let first start with a procedure for determining $m_{11},\ m_{10},\ m_{01}$ and $m_{00}$. When the first step is "success" $m_0 = 1$ then the possible outcomes could be "success" or "failure" so we add $w$ and $v$ to each column in the matrix. When the first step is "failure" $m_0 = 0$ then the possible outcomes could be "success" or "failure" so we add $y$ and $x$ to each column in the matrix. In the next step if we have "success" in the previous one which means $w$ or $y$ we add $w$ and $v$ to each column in the matrix. In case when we have in the previous step "failure" which means $v$ or $x$ we add $y$ and $x$ to each column in the matrix. The algorithm is given in Algorithm 1 box.

**Algorithm 1** Calculate matrix of all possible states of risky asset using non-recombinant first order Markov chain

```
mat ← (uw; uv; dx; dy)
n ← 10
for n = 3 → n do
    [nCol, nRow] = size(mat)
    for j = 1 → nCol do
        if mat(j,:) = w then
            lastStepCol ← [mat(j,:), w; mat(j,:), v]
        else if mat(j,:) = v then
            lastStepCol ← [mat(j,:), x; mat(j,:), y]
        else if mat(j,:) = x then
            lastStepCol ← [mat(j,:), w; mat(j,:), v]
        else if mat(j,:) = y then
            lastStepCol ← [mat(j,:), x; mat(j,:), y]
        end if
    end for
    mat ← [mat, lastStepCol]
end for
print mat
```

Using the those algorithms, we produce a list of all allowed **m** at a fixed depth $n$. Because we do not count the duplication of states, then a tree of depth $n$ will contain $2^n$ states.

Calculating the Markov tree is very difficult and needed high computational power but once we generated it we could use it many times to price many different options. For different options, $S0,\ u,\ d,\ v,\ w,\ x,$ and $y$ will be different, but the set of states will always be the same.

## Conflict of Interests

The author declares that there is no conflict of interests.


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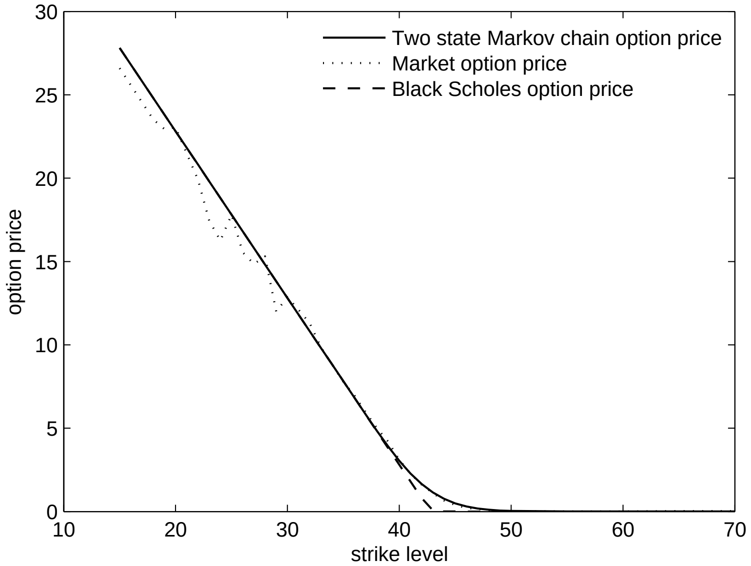


FIGURE 1. European call option on prices JPM under two state Markov chain model with non-recombinant tree tree, Black-Scholes model and market price. The current value S of the JPM is 42.81, the risk-free rate $r$ is 0.01% annually as a yield of 1-month Treasury bills. The dot line is a market option price for different strike prices. The dash line is a theoretical price by Black-Scholes model and the line is an option price under the Markov chain model with non recombinant-tree.

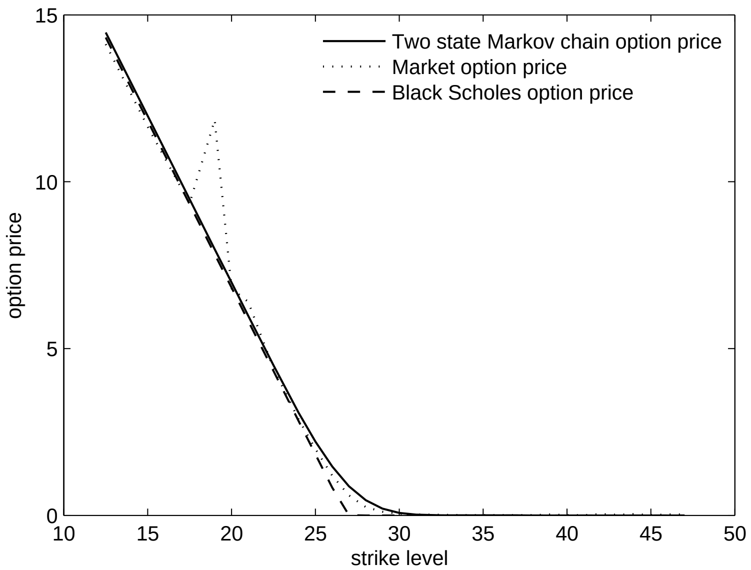


FIGURE 2. European call option prices on MSFT under two state Markov chain model with non-recombinant tree, Black-Scholes model and market price. The current value S of the JPM is 26.80, the risk-free rate $r$ is 0.01% annually as a yield of 1-month Treasury bills. The dot line is a market option price for different strike prices. The dash line is a theoretical price by Black-Scholes model and the line is an option price under the Markov chain model with non recombinant-tree.

TABLE 1. Estimated equivalent martingale measure (EMM) parameters and respective error estimates (06 June 2014)

| Company | APX | | | CSCO | | | GE | | | IBM | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Expiry date | 1m | 3m | 6m | 34 days | 3m | 6m | 1m | 3m | 6m | 1m | 3m | 6m |
| S | | 92.80 | | | 24.70 | | | 26.77 | | | 185.98 | |
| $\sigma$ | | 0.0121 | | | 0.0156 | | | 0.0112 | | | 0.0110 | |
| v | 1.0155 | 0.9667 | 1.0551 | 1.0131 | 0.9706 | 0.9647 | 0.9801 | 1.0231 | 0.9475 | 0.9934 | 1.9691 | 1.0012 |
| y | 1.0013 | 0.9983 | 0.9953 | 0.9910 | 1.0010 | 1.0076 | 1.0000 | 1.0004 | 1.0002 | 0.9898 | 1.9998 | 1.0012 |
| u | 1.0122 | 1.0122 | 1.0122 | 1.0157 | 1.0157 | 1.0157 | 1.0112 | 1.0112 | 1.0112 | 1.0110 | 1.0110 | 1.0110 |
| $AAE_{mod}$ | 0.2408 | 0.3381 | 0.1602 | 0.3640 | 0.3126 | 0.2613 | 0.3176 | 0.2176 | 0.4218 | 0.3789 | 0.2263 | 0.1899 |
| $AAE_{BS}$ | 0.2660 | 0.5245 | 0.1851 | 0.8335 | 0.9724 | 1.6281 | 0.7623 | 0.3768 | 0.5047 | 1.1562 | 0.2989 | 0.3117 |
| $APE_{mod}$ | 0.0921 | 0.2259 | 0.1034 | 0.0457 | 0.0914 | 0.0447 | 0.0375 | 0.0263 | 0.0722 | 0.0526 | 0.0422 | 0.0381 |
| $APE_{BS}$ | 0.0992 | 0.2460 | 0.1062 | 0.6480 | 0.1278 | 0.2005 | 0.0807 | 0.0467 | 0.0928 | 0.0896 | 0.0596 | 0.0509 |
| $RMSE_{mod}$ | 1.332 | 2.2891 | 2.7645 | 0.2050 | 0.4205 | 0.0886 | 0.1105 | 0.1001 | 0.4164 | 0.8366 | 0.6320 | 1.0604 |
| $RMSE_{BS}$ | 1.382 | 2.3953 | 2.7960 | 0.2428 | 4641 | 0.2904 | 0.1865 | 0.1562 | 0.4556 | 1.0994 | 0.8264 | 1.2384 |

The risk-free rate $r$ is 0.25% annually. An appropriate Treasury bond is selected 1-month Treasury bill. Days to expiry of 1m option is 34 days, 3m - 70 days and 6m - 133 days

TABLE 2. Estimated equivalent martingale measure (EMM) parameters and respective error estimates (06 June 2014)

| Company | INTC | | | JPM | | | KO | | | MMM | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Expiry date | 1m | 3m | 6m | 1m | 3m | 6m | 1m | 3m | 6m | 1m | 3m | 6m |
| S | | 27.66 | | | 56.63 | | | 40.89 | | | 143.71 | |
| $\sigma$ | | 0.0126 | | | 0.0146 | | | 0.0301 | | | 0.0091 | |
| v | 1.0266 | 0.9606 | 0.9717 | 1.0221 | 0.9783 | 0.9602 | 0.9992 | 1.0284 | 0.9580 | 1.0104 | 0.9583 | 0.9261 |
| y | 1.0001 | 1.0001 | 0.9860 | 1.0000 | 0.9999 | 1.0002 | 1.0000 | 1.0001 | 1.0004 | 1.0014 | 1.0003 | 1.0006 |
| u | 1.0128 | 1.0128 | 1.0128 | 1.0147 | 1.0147 | 1.0147 | 1.0306 | 1.0306 | 1.0306 | 1.0091 | 1.0091 | 1.0091 |
| $AAE_{mod}$ | 0.2799 | 0.4072 | 0.3253 | 1.0020 | 0.2029 | 0.5183 | 04776 | 0.2785 | 0.2790 | 0.3876 | 0.2176 | 0.2551 |
| $AAE_{BS}$ | 0.5696 | 0.7048 | 0.6408 | 3.2153 | 2.2775 | 0.6388 | 10.7481 | 15.9615 | 14.5118 | 0.6002 | 0.3864 | 0.2099 |
| $APE_{mod}$ | 0.1210 | 0.1656 | 0.0647 | 0.2094 | 0.0341 | 0.2144 | 0.0737 | 0.0721 | 0.0412 | 0.1007 | 0.0587 | 0.0599 |
| $APE_{BS}$ | 0.1523 | 0.1849 | 0.1865 | 0.3962 | 0.1809 | 0.3354 | 0.8795 | 0.5043 | 1.3920 | 0.1811 | 0.0664 | 0.0639 |
| $RMSE_{mod}$ | 0.5841 | 0.6736 | 0.1311 | 0.5224 | 0.1497 | 0.1805 | 0.1606 | 0.5606 | 0.1138 | 0.6704 | 1.2738 | 1.7679 |
| $RMSE_{BS}$ | 0.6084 | 0.6984 | 0.2432 | 0.8357 | 0.6262 | 0.3965 | 0.5679 | 1.8654 | 3.3806 | 0.9779 | 1.3713 | 2.0409 |

The risk-free rate $r$ is 0.25% annually. An appropriate Treasury bond is selected 1-month Treasury bill. Days to expiry of 1m option is 34 days, 3m - 70 days and 6m - 133 days

TABLE 3. Estimated equivalent martingale measure (EMM) parameters and respective error estimates (06 June 2014)

| Company | MRK | | | MSFT | | | PFE | | | PG | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Expiry date | 1m | 3m | 6m | 1m | 3m | 6m | 1m | 3m | 6m | 1m | 3m | 6m |
| S | | 58.10 | | | 41.21 | | | 29.76 | | | 80.10 | |
| $\sigma$ | | 0.0107 | | | 0.0142 | | | 0.0099 | | | 0.0091 | |
| v | 1.0119 | 1.0246 | 0.9663 | 1.0129 | 0.9702 | 0.9678 | 1.0129 | 0.9766 | 0.9479 | 1.0012 | 1.0110 | 0.9871 |
| y | 0.9991 | 1.0034 | 0.9989 | 0.9994 | 1.0053 | 1.0058 | 1.0085 | 0.9999 | 1.0002 | 1.0012 | 1.0110 | 0.9777 |
| u | 1.0108 | 1.0108 | 1.0108 | 1.0143 | 1.0143 | 1.0143 | 1.0099 | 1.0099 | 1.0099 | 1.0092 | 1.0092 | 1.0092 |
| $AAE_{mod}$ | 0.2590 | 0.0997 | 0.3205 | 0.2861 | 0.1396 | 0.2677 | 0.5037 | 0.3554 | 0.4304 | 0.3644 | 1.0157 | 0.0386 |
| $AAE_{BS}$ | 0.4706 | 0.2523 | 0.1318 | 0.7210 | 1.1086 | 0.9247 | 0.5882 | 0.2798 | 0.2971 | 0.4326 | 1.0106 | 0.4903 |
| $APE_{mod}$ | 0.1224 | 0.1617 | 0.1020 | 0.0921 | 0.0711 | 0.1534 | 0.0429 | 0.0324 | 0.0563 | 0.0479 | 0.0286 | 0.0124 |
| $APE_{BS}$ | 0.1403 | 0.2341 | 0.0979 | 0.1183 | 0.0781 | 0.2711 | 0.0575 | 0.0314 | 0.0576 | 0.0539 | 0.1301 | 0.0755 |
| $RMSE_{mod}$ | 1.9735 | 0.5267 | 0.7407 | 0.7642 | 0.2399 | 0.4821 | 0.0517 | 0.0912 | 0.2422 | 0.5846 | 0.1226 | 0.0665 |
| $RMSE_{BS}$ | 2.0039 | 0.6378 | 0.7567 | 0.8265 | 0.3333 | 0.6811 | 0.0688 | 0.1039 | 0.2566 | 0.6186 | 0.3752 | 0.4173 |

The risk-free rate $r$ is 0.25% annually. An appropriate Treasury bond is selected 1-month Treasury bill. Days to expiry of 1m option is 34 days, 3m - 70 days and 6m - 133 days

TABLE 4. Estimated equivalent martingale measure (EMM) parameters and respective error estimates (06 June 2014)

| Company | T | | | TRV | | | WMT | | | XOM | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Expiry date | 1m | 3m | 6m | 1m | 3m | 6m | 1m | 3m | 6m | 1m | 3m | 6m |
| S | | 35.10 | | | 97.97 | | | 77.32 | | | 100.55 | |
| $\sigma$ | | 0.0096 | | | 0.0093 | | | 0.0084 | | | 0.0090 | |
| v | 1.0000 | 0.9985 | 0.9983 | 0.9895 | 1.0623 | 0.9650 | 1.0000 | 1.0012 | 1.0110 | 1.0001 | 1.0110 | 1.0012 |
| y | 1.0427 | 0.9798 | 0.9769 | 0.9994 | 1.0002 | 0.9961 | 1.0102 | 1.0012 | 1.0110 | 1.0153 | 1.0110 | 1.0012 |
| u | 1.0096 | 1.0096 | 1.0096 | 1.0093 | 1.0093 | 1.0093 | 1.0090 | 1.0090 | 1.0090 | 1.0090 | 1.0090 | 1.0090 |
| $AAE_{mod}$ | 0.4471 | 0.2847 | 0.1316 | 0.1316 | 0.1384 | 0.2146 | 0.3445 | 0.1189 | 0.1488 | 0.2493 | 0.1283 | 0.3546 |
| $AAE_{BS}$ | 0.4429 | 0.3970 | 0.5355 | 0.5355 | 0.3525 | 0.3134 | 0.1945 | 0.3017 | 0.6584 | 0.3328 | 0.2564 | 0.2801 |
| $APE_{mod}$ | 0.0385 | 0.0659 | 0.0201 | 0.0201 | 0.0915 | 0.0252 | 0.0438 | 0.1040 | 0.0361 | 0.0366 | 0.0400 | 0.0586 |
| $APE_{BS}$ | 0.0452 | 0.0975 | 0.0421 | 0.0421 | 0.2015 | 0.0431 | 0.0732 | 0.1292 | 0.0545 | 0.0464 | 0.0584 | 0.0741 |
| $RMSE_{mod}$ | 0.1542 | 0.2435 | 0.2143 | 0.2143 | 0.7634 | 0.4156 | 0.2927 | 0.8015 | 0.5397 | 0.4937 | 0.1984 | 0.6504 |
| $RMSE_{BS}$ | 0.1863 | 0.2971 | 0.2778 | 0.4711 | 1.3304 | 0.6708 | 0.3762 | 0.8607 | 0.5951 | 0.5544 | 0.2799 | 0.8952 |

The risk-free rate $r$ is 0.25% annually. An appropriate Treasury bond is selected 1-month Treasury bill. Days to expiry of 1m option is 34 days, 3m - 70 days and 6m - 133 days

TABLE 5. Summarized error results

| Type of errors | 1m | 3m | 6m | all |
|---|---|---|---|---|
| $AEE_{mood}$ $vs.$ $AEE_{bs}$ | 90.00% | 86.87% | 73.33% | 83.4% |
| $APE_{mood}$ $vs.$ $APE_{bs}$ | 100.00% | 90.00% | 86.87% | 92.29% |
| $RMSE_{mood}$ $vs.$ $RMSE_{bs}$ | 100.00% | 93.33% | 100.00% | 97.77% |
| *Total model error vs. Total BS error* | 96.66% | 90.07% | 86.73% | 91.15% |

The percentage of option prices with maturity 1-month, 3-month and 6-month having results from errors show that, two-state Markov chain model with non-recombinant tree gets closer option value to the market price of option than the price calculated by the Black-Schole model.