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REDUCTION ERROR IN ASIAN OPTION PRICING BASED ON PARTITION MONTE CARLO METHOD

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Abstract. Monte Carlo simulation is the use of experiments with random numbers to evaluate mathematical expressions. The base experimental units are random numbers. The expressions may be definite integrals, systems of equations and financial engineering. In problems of moderate dimensions, quasi-Monte Carlo method usually provides better estimates than the Monte Carlo method. In this paper, we study Faure sequence(Faure sequence is low-discrepancy sequence), and introduce partition Monte Carlo and we employ to obtain significant improvement in Asian option price model.

Keywords: Monte Carlo, quasi-Monte Carlo, partition Monte Carlo, Faure sequence, Asian option price.

2010 AMS Subject Classification: 91G60.

1. Introduction

The Monte Carlo (MC) method has been introduced in finance in 1977, in the pioneering work [3]. In 1995, Paskov and Traub in [16] published a paper, in which they used quasi-Monte Carlo (QMC) methods to estimate the price of a collaterized mortgage obligation [6]. The problem they considered was in high dimensions (360) nevertheless, they obtained more accurate approximations with QMC methods than with the standard MC method [5]. Since then, many people have been looking at QMC methods has a promising alternative for pricing financial products [1,2,4,12,14,18,21]. Researchers study on QMC methods have also been very interested by these advances in computational finance because they provided convincing numerical results suggesting that QMC methods could do better than MC even in high dimensions, a task that was generally believed to be out of reach.

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Asian options are of particular importance for commodity products which have low trading volumes, since price manipulation is inhibited. Hence, the pricing of such options becomes one of the most interesting fields. Since there are no known closed form analytical solutions to arithmetic average Asian options, many numerical methods are applied.

This paper deals with pricing of geometric average Asian options with the help of Monte Carlo methods. We also investigate ways to improve the precision of the simulation estimates through the partition Monte Carlo technique.

2. Monte Carlo Method (MC)

Monte Carlo method is an analytical technique for solving a problem by performing a large number of trial runs, called simulations, and inferring a solution from the collective results of the trial runs. Monte Carlo integration is to use random points for the numerical evaluation of an integral [22].

$$I=\int_a^b f(x)dx \approx \frac{b-a}{N}\sum\nolimits_{i=1}^{N} f(x_i)$$

in this method I is approximate by taking random variables X_i and arithmetic averaging by contribution $f(x_i)$. In the special case of the above equation is integration bounded on $[0,1)$,

$$I=\int_0^1 f(x)dx \approx \frac{1}{N}\sum\nolimits_{i=1}^{N} f(x_i).$$

The base of Monte Carlo method is to generate independent identically distributed random variables on $(0,1)$, which the interval number uniformly distributed on $[0,1)$ to generate these distributed random variables, we employ the rand function on Matlab programming software.

3. Quasi-Monte Carlo Methods

As an alternative of the MC method for the general problem for which QMC methods have been proposed is multi-dimensional numerical integration. Hence, for the remainder of this section, we assume the problem under consideration is to evaluate

$$I=\int_0^1 f(u)du,$$

where *f* is a square-integrable function. Many problems in finance amount evaluate such integrals, as we discuss in Section 3. To approximate I, both MC and QMC proceed by choosing a point set $P_n = \{u_0, u_1, \dots, u_{n-1}\} \subset [0,1)^t$, and then the average value of f over P_n is computed, i.e., we get,

$$Q_n = \frac{1}{n}\sum_{i=0}^{n-1} f(u_i)$$

in the MC method, the points $u_0, u_1, \dots, u_{n-1}$ are independent and uniformly distributed over $[0,1)^t$. In practice, one uses a pseudorandom number generator to choose these points. The idea of QMC methods is to use a more regularly distributed point set, so that a better sampling of the function can be achieved. An important difference with MC is that the set P_n is typically deterministic when a QMC method is applied.

Quasi-random (deterministic) numbers are similar to random numbers but exhibit much more regularity. This makes them well-suited for numerical evaluation of multi-dimensional integrals. The main types of quasi-random sequences are encompassing Halton, Faure, Sobol, and Korobov sequences. In this section for achieving the goal of this paper Faure sequence and its structure will be introduced.

The main problem of the Faure sequence is that the differences between sample size and dimension are small. Faure sets points $\mathrm{b_j} = \mathrm{b}$ for $\mathrm{j} = 1, \dots, \mathrm{s}$ and uses the powers of the upper triangular Pascal matrix modulo b for the generator matrices. The $\mathrm{n^{th}}$ element of the Faure sequence is expressed as

$$X_n = (\varphi_b(P^0 n'), \varphi_b(P^1 n'), .., \varphi_b(P^{s-1} n'))$$

here $\varphi_b(n')$ is the radical inverse function in base b, and it is expressed as

$$\varphi_b(n') = \frac{n_0}{b} + \frac{n_1}{b^2} + \cdots + \frac{n_m}{b^{m+1}}$$

that $n = (n_0, n_1, \dots, n_m)^T$ is the digit vector of the b-adic representation of n. b is a prime number greater than or equal to the dimension s and P is the Pascal matrix modulo b whose (i, j)-element is equal to $\binom{j-1}{i-1}$ mod b. The matrix vector products $P^j n'$ for $j = 0, \dots, s-1$, are done in modulo b arithmetic. In generating quasi-random number generators for Monte Carlo computations, we may employ for sequence.

4. PMC method

Estimated results of Asian option price by the Monte Carlo and quasi-Monte Carlo are reliable, however we can significantly improve the results by managing and partitioning of the random numbers generated that is known as p-rand or PMC method.

The quasi random numbers for generating random number we can improve the uniform property of random number on [0,1] by partitioning the interval [0,1] to k sub intervals. Then, we independently generate the random numbers on each sub intervals. For example, if we generate 1000 random number on [0,1] by a random generator, in a PMC method we may generate 100 random number on each sub interval $[\frac{i-1}{10},\frac{i}{10}]$ independently, where $i = 1,2,\dots,10$. This method will control the uniformity of distributed random number on desired sub intervals $[\frac{i-1}{10},\frac{i}{10}]$ performed on [0,1]. It is expected to have more accuracy in results as we increase the number of partitioned sub intervals. In fact, we would like to examine this claim by investigating this idea on finance engineering in the numerical results section. For example we express algorithm and histogram PMC method.

Algorithm of the method(PMC):

Step 1. Choose N as the number of random numbers.

Step 2. Choose number of sample points.

Step 3. Choose N/K number of sub-interval's j, with $j = 1,2,\dots,K$, where K is the a number of subintervals.

Results are shown separately in two histograms.

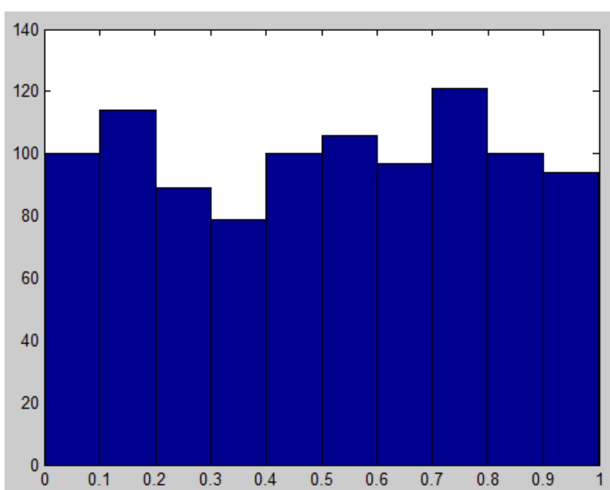

Fig 1. Histogram for N=1000, using the rand-function in Matlab programming software.

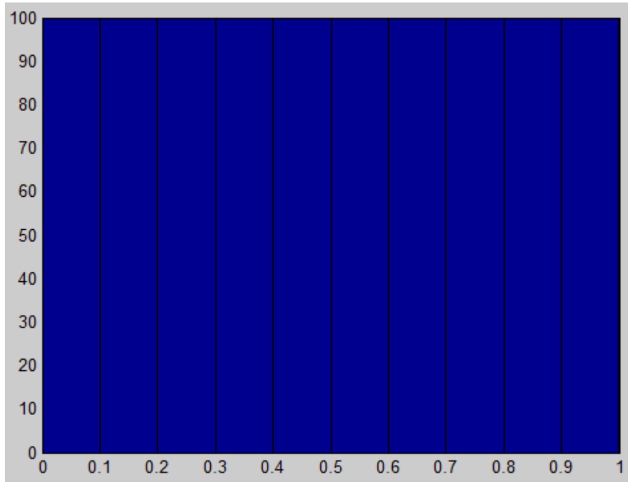

Fig 2. Histogram for N=1000, partitioning by use of the rand-function in Matlab programming software (p-rand).

The first histogram shows the numbers that are randomly generated. But, the second histogram is stand for showing the partitioned randomly generated numbers. Obviously, points discrepancy will be decreased by partitioning of the random numbers generated, and consequently results will be improved by reducing error.

The purpose of this article is to apply partition Monte Carlo method that improves the results of Asian option price. We compare these obtained results with other method's results by relative error and CI width (confidence interval width) in Asian option. In the next section, we discuss Asian option pricing.

5. Asian option pricing

An Asian option is a financial derivative whose payoff depends on the average price of the underlying asset. There are many varieties: Other than the usual call and put variations, there are:

- Fixed strike Asian options whose payoff is the difference (if positive) between the average price and a fixed strike price.
- Floating strike options whose payoff is the difference (if positive) between the final stock price and the average stock price.
- American versions of the Asian option (not discussed in this paper), which allow for early exercise (as opposed to the European version which can only be exercised at expiry).

There are also variations in terms of how the averaging is defined. The most important is whether the average is arithmetic or geometric (the former being far more common in practice). This distinction is important, not just in practice, but in terms of pricing and analyzing these options.

Another important distinction is whether the average is taken over a discrete set of observations, or a continuous set. The average may be a weighted average, with more weights being assigned to observations deemed to be more important. Finally, the averaging period may start at a future date (known as forward starting options).

There are two basic forms of Asian Option [20]:

An average price option is an option which at expiry pays the difference between the average value of the underlying during the life of the option (averaging period) and a fixed strike.

$$Call\ option\ payoff = C_{ave} = max(G_T - F, 0)$$

$$put\ option\ payoff = P_{ave} = max(F - G_T, 0)$$

where $G_T = \left(\prod_{j=0}^{m} S_j\right)^{1/(m+1)}$.

An average strike option is an option which at expiry pays the difference between the underlying market price and the strike which is an average of the underlying price over the specified averaging period [6]. In our work we will concentrate on the Average price option since it is much more common.

5.1. Simulation of Stock Prices

Given a stock paying continuous dividends at the rate of, and a continuously compounding risk-free interest rate of r, paths of stock prices can be simulated using Equation (1) for the stock price by the following equation in [21]. Which will be resulted in

$$S_t = S_0 exp((r - \sigma^2/2)dt + \sigma\sqrt{dt}Z \qquad (1)$$

where Z has a standard normal distribution. Let S_t denote the price of the stock at time $t \in [0, T]$. Starting with the initial stock price S_0, we simulate the stock price at the next time value $dt = T/m$, by taking a realization Z_j of a standard normal random variable and evaluating the right-hand-side of the equation. This process is repeated using the previous price as the new initial price. For example

$$S_{j+1} = S_j exp\left(\left(r - \frac{\sigma^2}{2}\right)dt + \sigma\sqrt{dt}Z_j\right), \qquad j = 0,1,\dots,m$$

which that Z_j for $j = 0,1,\dots,m$ are standard normal distribution [20].

Algorithm:

The Monte Carlo (MC) implementation to price this derivative:

Step1. Find each i^{th} path we find;

$$G_T^{(i)} = \left(\prod_{j=0}^{m} S_j\right)^{1/(m+1)}, \qquad dt = \frac{T}{m} \,\&\, i = 1,\dots,N \times \text{M}.$$

Step2. Calculate payoff of the i^{th} path. For example, for an average price call the payoff is

$$C_{ave}^{(i)} = max\left(0, G_T^{(i)} - F\right) \quad , \ for \ i = 1,\dots,N \times M \ .$$

Step3. Discount this value by the risk-free rate to get the price of the option:

$$\hat{C}_{ave}^{(i)} = e^{-rT}.C_{ave}^{(i)}.$$

Step4. Take an average of those payoffs:

$$\hat{C}_{ave} = \frac{1}{N\times M}.\sum_{i=1}^{N\times M} \hat{C}_{ave}^{(i)}.$$

Step5. Compute the relative error:

Std $(\hat{C}_{ave})/(\hat{C}_{ave} \times \sqrt{N \times M})$.(Std is standard division).

5.2 Pricing

Here we consider a problem from computational finance: pricing of geometric Asian call options. In simulation, we generate a sequence of asset prices $S_0, S_1, .., S_m$ that is subject to an Ito process $dS = \mu Sdt + \sigma SdX$ where t is time, μ and σ are the drift and volatility of the underlying, respectively, and $X = X(t)_t$ is a standard Brownian motion [6]. Let $h(S_0, S_1, .., S_m) = max(G_T - F, 0)$ be a call option payoff that $G_T = \left(\prod_{j=0}^{m} S_j\right)^{1/(m+1)}$ is the geometric average of the asset prices, and F is the strike price. The price of the Asian call option is the expected value

$$\theta = E[e^{-rT}h(S_0, S_1, .., S_m)], \tag{2}$$

which is estimated by simulation. In this expression r is the risk-free interest rate and T is the expiration time, i.e., the time when we observe the final price S_m. To calculate the expected value, the probability distribution for average should be known [17].

If the average price found as geometrical average then there are analytical formulas for valuing European average price option (since the price of the asset is assumed to be lognormally distributed and geometric average of a set of lognormally distributed variables is also lognormal). The geometric average is defined as:

$$G_T = \left(\prod_{j=0}^{m} S_j\right)^{1/(m+1)}.$$

Therefore the price of the geometric Asian call option is given by a modified Black-Scholes formula [5],

$$C_{G\_Asian} = e^{-rT}.\,(exp(\,a+b)/2).\,N(x) - F.\,N(x-\sqrt{b}), \tag{3}$$

where

$$a = ln(G_t) + \frac{m-k}{m}\left[ln(s_0) + \left(\mu - \frac{\sigma^2}{2}\right)(t_{k+1} - t) + \frac{1}{2}.\left(\mu - \frac{\sigma^2}{2}\right)(T - \ t_{k+1})\right],$$

$$b = \left(\frac{m-k}{m}\right)^2.\,\sigma^2(t_{k+1} - t) + \frac{\sigma^2(T - \ t_{k+1})}{6m^2}(m-k)(2(m-k)-1),$$

$$x = \frac{a - ln(F) + b}{\sqrt{b}},$$

where G_t is the current geometric average, Fis the last known fixingand in this expression under the risk-free rate $\mu = r$.

6. Numerical Results

In this section, we implemented PMC method to the financial engineering and also significant improvement will be seen in Asian call option price. Criterion like CI Width (confidence interval width) to show the uncertainty in the price, and any mispricing on this scale could result in an arbitrage opportunity and other citation is relative error. We estimated the option price using MC, PMC and QMC methods.

Let $S_0 = 40$, $T = 4/12$, $F = 35$, $\sigma = 0.2$, $\mu = r = 0.07$. Where μ and σ are the drift and volatility of the underlying, respectively, leading to an exact price of 5.3187.

f*igs.3, 4* and *fig.5, 6* displays the results when $X = N \times M$ the number of points simulation increases. That in the MC and PMC method M, the number of replications, is equal 1 and N is increased. And in QMC method the number of replications, M is chosen fix 10.

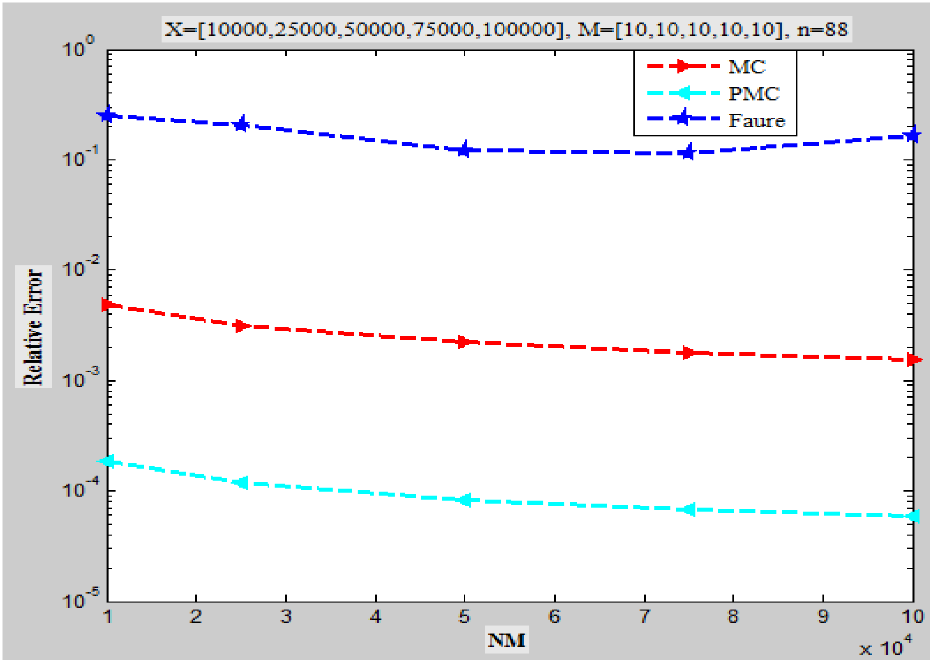

Fig 3. Relative error for pricing of an Asian call option, *with* $X = N \times M$ number of points. $N \times M$ increases (M is fix at 10, we only increase N) M is the number of replicationsand n = 88, n is trading days.

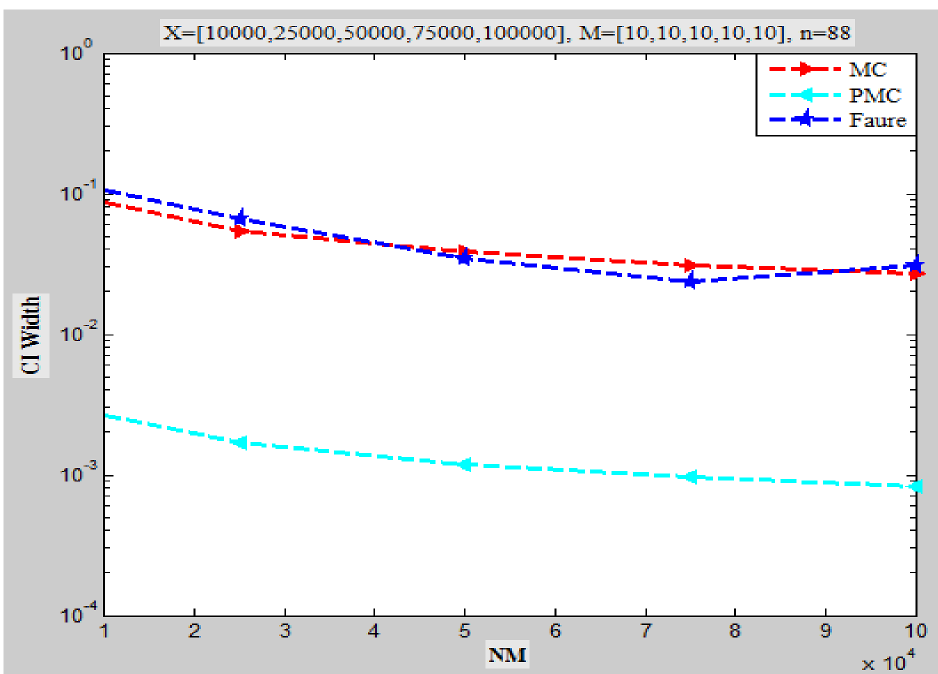

Fig 4. CI Width for pricing of an Asian call option, with X = N × M number of points. N × M increases (M is fix at 10, we only increase N) M is the number of replications and n = 88, n is trading days.

Table.1

	MC	MC	MC	MC	MC
X=N×M	10000	25000	50000	75000	1000000
Relative Error	0.004896	0.003100	0.002206	0.001790	0.001550
CI Width	0.085560	0.054141	0.038253	0.031255	0.027078

Table.2

	PMC	PMC	PMC	PMC	PMC
X=N×M	10000	25000	50000	75000	100000
Relative Error	0.000184	0.000118	0.000082	0.000068	0.000058
CI Width	0.002621	0.001674	0.001172	0.000961	0.000831

Table.3

	Faure	Faure	Faure	Faure	Faure
X=N×M	10000	25000	50000	75000	100000
Relative Error	0.252332	0.203729	0.120468	0.116130	0.166097
CI Width	0.002621	0.001674	0.001172	0.000961	0.000831

With $\mathrm{X} = 10000$, $\mathrm{X} = 25000$, $\mathrm{X} = 50000$, $\mathrm{X} = 75000, \mathrm{X} = 100000$ we obtain a typical estimate of Relative Error and CI Width for MC (*table.1*), PMC(*table.2*) and QMC(*table.3*), also we saw that $\mathrm{X} = 10^4$ relative error of the PMC 0.000184% and for $\mathrm{X} = 25 \times 10^3$ relative error of the PMC 0.000118% and also for $\mathrm{X} = 5 \times 10^4$ relative error of the PMC, 0.000082% and also for $\mathrm{X} = 75 \times 10^3$ and also $\mathrm{X} = 5 \times 10^5$ relative errors respectively are 0.000068% and 0.000058% it is plainly visible that PMC relative error is less of QMC and MC. Also for PMC CI Width (*table.2*) with $\mathrm{X} = 10^4$ there is about $\$0.002621$ uncertainly in the price, as the amount of X is increased to 25×10^3 there is about $\$0.001674$ uncertainly in the price, and for $\mathrm{X} = 5 \times 10^4,\ 75 \times 10^3,\ 10^5$ respectively uncertainly is equal $\$0.001172$, $\$0.000961$ and $\$0.000831$ vividly see that any mispricing on this scale could result in an arbitrage opportunity.

According to *fig.3* and *fig.4* the same expression is satisfied in *fig.5* and *fig.6* but in this figures (*fig.5, 6*) $X = [10^5, 25 \times 10^4, 5 \times 10^5, 75 \times 10^4, 10^6]$ and $M = [10,10,10,10,10,10]$ and it is plainly visible that the efficiency of this method is bright and higher than some applicable methods such as MC and QMC.

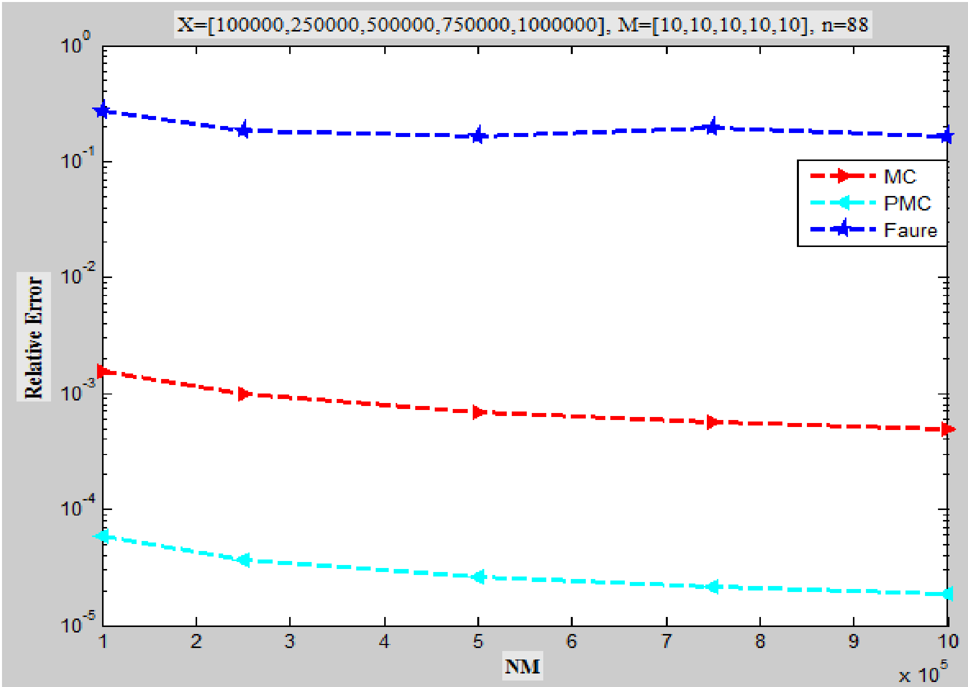

Fig 5. Relative error for pricing of an Asian call option, *with* $X = N \times M$ number of points. $N \times M$ increases (M is fix at 10, we only increase N) M is the number of replicationsand n = 88, n is trading days.

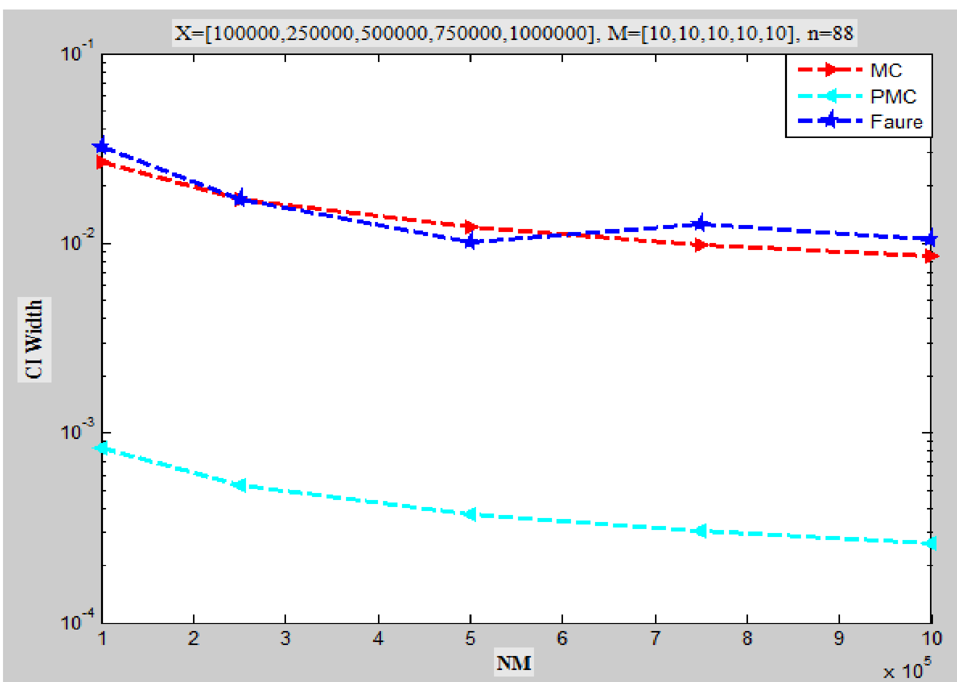

Fig 6. CI Width for pricing of an Asian call option, with X = N × M number of points. N × M increases (M is fix at 10, we only increase N) M is the number of replications and n = 88, n is trading days.

Conflict of Interests

The author declares that there is no conflict of interests.

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