Available online at http://scik.org
Math. Finance Lett. 2015, 2015:5
ISSN 2051-2929

PRICING AND HEDGING OF BEST OF ASSET OPTIONS, A MALLIAVIN CALCULUS APPROACH

A.O. AKEJU*, E.O. AYOOLA

Department of Mathematics, University of Ibadan, Nigeria

Abstract. In this paper, we developed a formulation for pricing and Hedging of Rainbow Option and in particular the Best of Asset Option with pay-off $\max(S_1, S_2, ...S_n, K)$.Rainbow option is a class of options that involves multiple assets and the behaviour of the underlying determine the specific type of the Rainbow option in question. In this study, we consider a Best of Asset type of Rainbow option with Pay-off given as max $(S_1, S_2, - - -S_n, K)$. Here, we make use of the Malliavin Calculus and the Clack Ocone formula to formulate the Price and the Hedging strategy in closed form.The price of the Best of Asset option will be determined from the Clark-Haussmann Ocone CHO formula as the discounted expectation of the pay-off $f(w)$ while the hedging portfolio will be obtained from the integrant in the Martingale representation theorem set up of the Payoff.The integrant involves the Malliavin derivative of the pay-off and its market price of risk and in the case that the latter is time -dependent, it reduces to the discounted expectation of the malliavin derivative of $f(w)$ conditioned with respect to the filtration.

Keywords: Rainbow Option; Malliavin Calculus; Clack Ocone Formula.

2010 AMS Subject Classification: 46F25, 60H07, 60H30.

1. Introduction

*Corresponding author

Received November 13, 2014

This work is developed in line with the works of Margable [5], on pricing of exchange Option which allows one to exchange an asset for another and the works of Mataranvura [7], where the price and the hedging of an exchange option was provided with the help of the CHO formula.

A fundamental question is,how should this Option be priced. In a standard Black-Sholes formula,the unique arbitrage free price is given by the discounted expectation of the Pay-off under the unique risk-Neutral measure.Although ,in most literature,effort has been dedicated to pricing of contingent claim, perhaps due to it important to market operator and traders, but less attention has been paid to hedging which actually help the market players in decision making. A solution of the hedging problems determine an arbitrage free price via the initial value of the hedging strategy.

The MRT expresses every square integrable martingale as a sum of a pre-visible process and an Ito integral.The advantage of the CHO [8] is that it allows the use of malliavin derivative for computing the hedging portfolio.The malliavin derivative is an alternative mathematical operation to delta hedging approach. Delta hedging approach has a set back because the derivative of certain payoff that are not differentiable everywhere are not obtainable,but the malliavin derivative ensure the differentiability of such functions.

In [2], a recursive procedure for pricing and hedging European basket and rainbow options on N assets was developed with a payoff $w[f(s_t,K)]^+$ where s_t is a vector of N assets,f is a function representing the payoff with $w=+1$ for a call and $w=-1$ for a put. If for instance,$N=3$,then $s_t=(s_1,s_2,s_3)$ and $f(s_t,k)=(\theta_1 s_1+\theta_2 s_2+\theta_3 s_3)-K$ for the basket option with weights $(\theta_1,\theta_2,\theta_3)$ and $f(s_t,K)=\max(s_1,s_2,s_3)-K$ for a rainbow option .The procedure involves the decomposition of the basket or rainbow payoff function into sum of exchange option payoff function.

This recursive approach has several advantages over those already developed.Firstly,the underlying assets price may follow heterogeneous log normal processes.Some asset price could follow mean -reverting processes whilst others follow standard log normal processes. Secondly, the frame work provides a convention for selecting the implied volatilities vanilla option on the individual underlying asset that are used to price the basket option .This yield volatility skew consistent prices. Thirdly,the approach derived analytic approximation for multi-asset option

Greeks,and unlike other approaches, these Greeks will be influenced by the individual asset price volatilities and correlations. Hence, hedge ratios are consistent with the individual asset implied volatility and implied correlation skews. These however are not without limitations.

The work of [5] initiated the study of determining the value of an option which exchange one asset for another explicitly. This is today known as Margrabe option. In [7], it is provided a malliavin calculus approach for computing both the price and hedging portfolio of an exchange option with two underlying assets using the Clark Haussmann-Ocone (CHO) formula whose application in the generalized form was studied by [4]. This techniques was also considered in [8] but for Asian type of option in explicit form.

2. Review of Malliavin calculus

Here, we shall consider some features of the malliavin calculus as it relate to our work.We consider the notion of differentiability within the family of random variables that are equal to functions of independent increment of Brownian motion.

Consider a real Brownian motion W on the probability space (Ω, σ, μ) endowed with the Brownian filtration

$$\beta^{\omega} = \beta_t^{\omega}, \qquad t\varepsilon[0,T].$$

Let $\Delta_n^k = W_t^k - W_t^{k-1}$ be the kth increment of the Brownian motion for $k = 1, 2 - - - 2^n$.

Suppose $T = 1$ and for $n\varepsilon N$,

$$t_n^k = \frac{k}{2^n}, \quad k = 0, 1 \ldots 2^n$$

is the $(k+1)th$ element of the nth order dyadic partition of the interval [0,T],then we write $I_n^k = [t_n^{k-1}, t_n^k]$ as the kth interval of the partition.

We denote $\Delta_n = (\Delta_n^1, - - - \Delta_n^{2n})$, the $\mathbb{R}^{2n}$ vector of the nth order Brownian increment by C_{pol}^{∞} (the family of smooth function that together with their derivative of any order have at most polynomial growth).

Definition 2.1. [1] For every $X = \varphi(\Delta_n) \in S$,the stochastic derivative of X at time t is defined on S by $D_t X = \dfrac{\varphi(\Delta_n)}{x_n^k n(t)}$,where S is a set of Cylindrical functional. The definition is well posed i.e it is independent of n,if n,m $\in N$, then

$$X = \varphi_n(\Delta_n) = \varphi_m(\Delta_m) \in S \tag{1}$$

and

$$D_t(X) = \frac{\partial \varphi_n(\Delta_n)}{\partial x_n^k n(t)} = D_t X = \frac{\partial \varphi_m(\Delta_m)}{\partial x_m^k m(t)}.$$

If we endow S with the norm, then,

$$\|X\|_{1,2} = E(X^2)^{\frac{1}{2}} + E[\int_0^T (D^s X)^2 ds]^{\frac{1}{2}}$$

=

$$\|X\|_{L^2(\varphi)} + \|DX\|_{L^2([0,T],\varphi)}.$$

In what follow, we concentrate on the Hilbert Space $D^{1,2}$ which will be relevant space for all of our computations.

Definition 2.2. The space $D^{1,2}$ of the Malliavin - differentiation random variables is the closure of S with respect to the norm $\|.\|_{1,2}$.In other words, $X \in D^{1,2}$ iff $\exists$ a sequence $(X_n) \in S$ such that
1) $X = \lim_{n\to\infty} X_n \in L^2(\Omega)$.
2) The $\lim_{n\to\infty} DX_n \in L^2([0,T],\Omega)$.
Then the Malliavin derivative of X is defined as $DX = \lim_{n\to\infty} DX_n \in L^2([0,T]X\Omega)$. The differential operator D is linear but not bounded (reference) i.e., sup $\frac{\|DX\|_{L^2}}{\|X\|_{L^2}} = +\infty$.

If X ,Y $\in D^{1,2}$, then the product XY in general is not square integrable and so,it does not belongs to $D^{1,2}$. It is therefore worth while to use instead of $D^{1,2}$, the slightly smaller but closed under product.
$D^{1,\infty} = \cap_{p\geq 2} D^{1,p}$, where $D^{1,p}$ is the closure of S with respect to the norm

$$||X||_{1,p} = ||X||_{L^p}(\Omega) + \|DX\|_{L^p}([0,T],\Omega).$$

We observe (by generalisation) that $X \varepsilon D^1, p$ iff $\exists$ a sequence $(X_n \in S$ such that
1) $X = \lim_{n\to\infty} X_n \in L^p(\Omega)$. 2) The $\lim_{n\to\infty} DX_n \in L^p([0,T],\Omega)$. If $p \leq q$, by Holder's inequality, we get $||.||_{L^p}([0,T]\mathtt{X}\Omega) \leq T^{\frac{q-p}{pq}} \|.\|_{L^p}([0,T]\mathtt{X}\Omega)$ and so $D^{1,p} \leq D^{1,q}$.
In particular for every $X \in D^{1,p}$ with $p \geq 2$ and an approximating sequence (X_n) in L^p, we have $\lim_{n\to\infty} DX_n = DX$ in $L^2([0,T]\mathtt{X}\Omega)$.

Proposition 2.3. [1] Let $\varphi^1 \in C^{\infty}_{pol}(\mathbb{R})$, then
i)If $X \in D^{1,\infty}$, then $\varphi(X) \in D^{1,\infty}$ and $D\varphi(X) = \varphi^{'}(X)DX$
ii)If $X \in D^{1,2}$ and $\varphi, \varphi^{'}$ are bounded, then $\varphi(X) \in D^{1,2}$.

Proof If $X \in S, \varphi \in C^{'}$ and both φ and it first order derivative are bounded, then $\varphi(X) \in S$. If $X \in D^{1,2}$, then $\exists$ a sequence $(X_n) \in$ converging to $X \in L_2(\Omega)$ and such that (DX_n) converges to $DX \in L^2([0,T]\text{x}\Omega)$, then by (dominated convergence theorem), $\varphi(X_n)$ tends to $\varphi(X) \in L^2(\Omega)$.

Furthermore, $D\varphi(X_n) = \varphi^{'}(X_n)DX_n$ and $||\varphi^{'}(X_n)DX_n - \varphi^{'}(X)DX||_{L^2} \leq I_1 + I_2$ where

$$I_1 = ||(\varphi^{'}(X_n) - \varphi^{'}(X))DX||_{L^2} \to 0, n \to \infty$$

(1); it suffice that $\varphi \in C^{'}$ and that both φ and its first order derivative have at most polynomial growth by dominated convergence theorem and

$$I_2 = \|\varphi^{'}(X_n)(DX - DX_n)\|_{L^2} \to 0, n \to \infty$$

Since (DX_n) converges to DX and $\varphi^{'}$ is bounded.

3. Pricing and hedging in a complete market

Let $(\Omega, P, \mu, \mathscr{F}_t)$ be a filtered probability space endowed with the filtration $\mathscr{F}_t$ generated by the Brownian motion.The filtration represent the flow of available information to the trader concerning the assets in the market at any time $t > 0$ .We define the prices of the assets on this filtered probability space.Here ,we consider two types of asset,the first is a risk-less asset with differential

$$dX(t) = \mu(t)X(t)dt \tag{2}$$

satisfying the assumption of existence and uniqueness of solution $X(t)$ with $X_0(0) = x_0$ been the price of the asset at time $t = 0$, where $\mu(t)$ is the risk-less interest rate which is consider to be a constant.It should be noted however that $\mu(t)$ can also varies with time as examined in [3]. The second asset are risky assets in form of stocks, foreign exchange, crude resources e.t.c. The price $X_i(t)$ of the assets i is given by the differential

$$dX_i(t) = \mu(t,w)dt + \sum_{j=1}^{n} \sigma_{i,j}(t,w)dW_j(t) \tag{3}$$

$X_i(0) = x_0 \quad i = 1,2---n$, where $\mu(t,w)$ is the rate of return of each asset and $\sigma_{i,j}$ is the volatility coefficient of the Brownian motion W_j in security i.

If we assume $\mu_i(t,w) = \begin{bmatrix} \mu_1 \\ . \\ . \\ . \\ \mu_n \end{bmatrix}$ be the vector of the rate of return for the asset i and that the matrix

$\sigma(t,w) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & - & - & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & - & - & \sigma_{2n} \\ . & & & & \\ . & & & & \\ \sigma_{n1} & \sigma_{n2} & - & - & \sigma_{nn} \end{bmatrix} \geq 0$ be the matrix of coefficient of volatility. If

$X(t) = \begin{bmatrix} X_1(t) \\ . \\ . \\ . \\ X_n(t) \end{bmatrix}$,then for conveniency, we can write (3) as

$$dX(t) = \mu(t)dt + \sigma(t)dW(t), \quad X(0) = x_0, \quad t \in [0,T]. \tag{4}$$

Let $\Theta(t) = \Theta(t,w) = (\Theta_0(t,w),\Theta_1(t,w)---\Theta_n(t,w))$ represent the investor's holding at any time $t \in [0,T]$,where $\Theta_i(t,w)$ is the number of units of asset i that the investor can hold.Any investor that select a portfolio consisting of n assets will have to define the proportion of his wealth that he must invest in each of the n assets.The holder of a portfolio Θ may decide to short his position at any time $t \in [0,t]$ and then bank plus the trading gains up to the date of shortening his position.

In complete market setting,where we assume that the portfolio is self financing ,so that the value of this portfolio at time $t \in [0,t]$ is given by

$$V^{\Theta}(t) = V^{\Theta}(t,w) = V(0) + \int_0^t \Theta_0 dX_0(s) + \sum_{i=1}^{n-1} \int_0^t \Theta_i(s) dX_i(s)$$

The portfolio Θ is called admissible if it is self financing and the value process $V^{\Theta}(t) : t \in [0,T]$ is bounded below. If we write the value of the portfolio $V^{\Theta}(t)$ as

$$V^{\Theta}(t) = \sum_{j=0}^{n-1} \Theta_j(t) X_j(t)$$

and assuming that the portfolio is self financing and admissible,then if σ is invertible,we have

$$dV^{\Theta}(t) = \rho(t)V^{\Theta}(t)dt + \Gamma(t)\sigma[\sigma^{-1}(\alpha - \rho I)dt + dB(t)],$$

where $\Gamma(t) = \begin{bmatrix} \Gamma_1 \\ . \\ . \\ . \\ \Gamma_n \end{bmatrix}$. If we let $u = \sigma^{-1}(\alpha - \rho I) = \begin{bmatrix} u_1 \\ . \\ . \\ . \\ u_n \end{bmatrix}$, where $X(t)$ is the vector of stock prices,and if we further assume that u satisfies the Novikov condition then by the Girsanov theorem $\widehat{B(t)} \in [0,T]$ given by $\widehat{B(t)} = \mu dt + dB(t)$ is a Brownian Vector with respect to the probability measure Q given by

$$dQ(w) = exp(-\int_0^T u(s)dB(s) - \frac{1}{2}\int_0^T \|u(s)\|^2 ds)dP(w)$$

where $||.||$ is the usual norm in $\mathbb{R}^n$. Then

$$dV^{\Theta}(t) = \rho(t)V^{\Theta}(t)dt + \Gamma(t)\sigma dB(t).$$

Solving for V^{Θ}, we get

$$exp - \int_0^T \rho(s)ds V^{\Theta} = V^{\Theta}(0) + \int_0^T e^{-\int_0^T \rho(s)} \Gamma(t)\sigma \widehat{dB}(t)$$

$$e^{\rho T} V^{\Theta}(T) = V^{\Theta}(0) + \int_0^T e^{-\rho t} \Gamma(t)\sigma \widehat{dB}(t).$$

This is a particular version of the Martingale Representation Theorem as applied to a particular square Integrable martingale $f(w) = e^{-\rho T} V^{\Theta}(T)$. It is this martingale Representation theorem which the CHO formula relies on.

In what follow, we state the Clack- Ocone -Formula and consequently, the generalized Clark-Ocone Haussmann formula without proof.

The martingale representation theorem asserts that for every $X \in L^2(\Omega, \mathscr{F}_t)$, there exist $u \in \mathbb{L}^2$ such that $X = E(X) + \int_0^T u_s dW_s$. If X is malliavin differentiable,we are able to obtain the expression of u such that $u_t \in D^{1,2}$ for every t and we have $D_t X = u_t + \int_0^T D_t u_s dW_s$ and so taking the conditional expectation,we can conclude that $E(D_t X|F_t) = u_t$. The Clack- Ocone formula is the main link between Hedging and Malliavin calculus and we state it as proposition as follow:

Proposition 3.1. (Clack -Ocone Formula) [1]. If X $\in D^{1,2}$,then

$$X = E(X) + \int_0^T E[(D_t X|F_t)]dW_t$$

One immediate consequence of the Clark Ocone Formula is that ,If $X \in D^{1,2}$ and $DX = 0$, then X is constant a.s. The financial interpretation and implication of the Clack -Ocone Formula is as follow. Suppose $X \in L^2(\Omega, \mathscr{F}_t)$ is the payoff of an European option on an asset S,with the dynamics of the discounted price under the EMM given as $d\widehat{S_t} = \sigma_t S_t dW_t$, then if (α, β) is a replicating strategy for the option, we have,

$$\widehat{X} = E(\widehat{X}) + \int_0^T \alpha_t d\widehat{S_t} = E(\widehat{X}) + \int_0^T \alpha_t \sigma_t \widehat{S_t} dW_t.$$

By the Clack-Ocone formula, we have

$$\widehat{X} = E(\widehat{X}) + \int_0^T E[(D_t X|\mathscr{F}_t)]dW_t$$

so we obtain the expression of the replicating strategy,

$$\alpha_t = \frac{E[(D_t X|\mathscr{F}_t)]}{\sigma_t \widehat{S_t}}, t \in [0,T].$$

This can be extended to multi- asset regime with

$$dS_t^i = \sigma_t^i S_t^i dW_t^i, i = 1,2,---n$$

so that

$$\alpha_t^i = \frac{E[(D_t X^i|F_t)]}{\sigma_t^i \widehat{S}_t^i}, \quad \forall i = 1,2---n.$$

Theorem 3.2. (The Generalized Clack -Ocone Haussmann Formula)[4] Suppose that $X \in \mathbb{D}_{1,2}$ and assume that the following conditions holds

(1) $E_Q[||X||_{L^2_Q}] \leq \infty$.

(2) $E_Q[\int_0^T ||D_t X||^2_{L^2_Q} dt] \leq \infty$.

(3) $E_Q[||X||^2_L(Q). \int_0^T (\int_0^T D_t U(s,w)dB(s) + \int_0^T D_t U(s,w).U(s,w))^2 dt] \leq \infty$, then

$$F(w) = E_Q[X] + \int_0^T E_Q[(D_t X - X \int_0^T D_t U(s,w)dB(s)F_t dB(t))|\mathscr{F}_t]dB(t),$$

where $U(s,w)$ is the Girsanov kernel,Q is the equivalent martingale measure and $\widehat{B}(t) = \widehat{B}(t.w)$ is a Brownian motion with respect to Q.

By letting $G(w) = e^{-\rho T} F(w)$ and applying the generalized CHO formula to G, we have

$$G(w) = E_Q[G] + \int_0^T E_Q[(D_t G - G \int_t^T [D_t U(s,w)d\widehat{B}(s)F_t d\widehat{B}(t)/\mathbb{F}_t]dB(t),$$

where D_t represent the Malliavin derivative.By the martingale representation theorem,we get

$$V(0) = V^Q(0) = E_Q[G]$$

and

$$e^{-\rho t}\Gamma(t)\sigma = E_Q[(D_t G - G \int_t^T [D_t U(s,w)dB(s)/F_t dB(t))/\mathbb{F}_t],$$

where

$$\Gamma(t) = e^{-\rho(T-t}\sigma^{-1} E_Q[(D_t G - G \int_t^T [D_t U(s,w)dB(s)/F_t dB(t))/F_t].$$

This gives the explicit number of unit of stocks.The holding $\Theta_i(t)$ is the Bank account which can be found from the self financing condition.

The implication of these result is that ,in a complete market,every contingent claim with pay-off $f(w)$ is attainable by the portfolio of stock and bonds.Therefore $V(0)$,the initial value of a self financing portfolio equals the price of such derivative since $F(t) = V(t)$.It then shows that the zero price of such a contingent claim is the discounted expectation of the pay-off.

4. The n-dimensional market model and transformation theorem

Suppose that a Portfolio consist of n- underlying assets mainly of risky securities like the Best Of Asset option.If we consider specifically the case of European Best of Asset option which have it prices at time t as $X_1(t), X_2(t), ---, X_n(t)$ with a risk-less Bank account which has it price as

$$dX_0(t) = \rho(t)X_0(t)dt.$$

Under the assumption of existence of a unique solution $X_0(t)$. where $\rho(t)$ is the interest rate.The risky securities is given as

$$X_i(t) = X_i(0)e^{((\alpha_i - \frac{1}{2}\sum_{j=1}^{n}\sigma_{i,j}^2)t + \sum_{j=1}^{n}\sigma_{i,j}B_j)} \quad i = 1, 2...n,$$

where $B_j(t), j = 1, 2, ---n$ is a standard Brownian motion. Suppose that the investor observe that at time $T > 0$ $P(X_1(t) > X_2(t) >, ---, X_n(t)) > 0,$ then the Payoff of the option becomes

$$f(w) = max(X_i(t) - X_j(t))^+, i \neq j.$$

We intend to determine the price and the Hedging portfolio of this option by using the generalized CHO formula. The Girsanov change of measure for this set up can be easily done by letting $\sigma.U = \alpha - \rho I$, where $I = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}$, $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \ldots \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \ldots \sigma_{2n} \\ . & . & \ldots \\ . & . & \ldots \\ . & . & \ldots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} \ldots \sigma_{nn} \end{bmatrix}$, $U = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_n \end{bmatrix}$, $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_n \end{bmatrix}$

With constant coefficient,U satisfies the Novikov conditions, so that the probability measure Q defined by $dQ(w) = M(T)dP(w)$ is equivalent to P and $e^{-\rho t}X_i(t)$ is a martingale with respect to Q.

$$M(t) = e^{(\sum_{j=1}^{n} U_j B_j - \frac{1}{2}\sum_{j=1}^{n} U_j^2)}$$

is a P- Martingale. With respect to Q, price X_i is

$$X_i(t) = X_i(0)e^{((\rho_i - \frac{1}{2}\sum_{j=1}^{n}\sigma_{i,j}^2)t + \sum_{j=1}^{n}\sigma_{i,j}B_j)}$$

so that

$$e^{-\rho t}X_i(t) = \widehat{X}_i(t) = X_i(0)e^{(\sum_{j=1}^{n}\sigma_{i,j}\widehat{B}_j - \frac{1}{2}\sum_{j=1}^{n}\sigma_{i,j}^2)}$$

$i = 1, 2, -, -, -n$ is a Q -Martingale. We assume that σ is invertible so that the market is complete.Suppose we choose a self financing portfolio $\Theta = (\Theta_0(t), \Theta_1(t), - - -\Theta_n(t))^{Tr}$,which is also admissible ,then the discounted value of the portfolio at any time $t < T$ is given by

$$e^{-\rho t}V^{\Theta}(t) = V^{\Theta}(0) + \int_0^t e^{-\rho s}\Gamma(s)d\widehat{B}(s),$$

where $\Gamma(s) = (\Theta_1(t), \Theta_2(t), - - -\Theta_n(t))^{Tr}$. From CHO formula, we note that for any contingent claim, $f(w) = V^{\Theta}(T)$, we get

$$V^{\Theta}(T) = E_{\Theta}[e^{-\rho t}f(w)]$$

and

$$\Gamma(t) = e^{-\rho(T-t)}\sigma^{-1}E_{\Theta}[D_t f(w)|F_t],$$

where, σ^{-1} is the inverse of σ such that the determinant of σ, $\Delta \neq 0$,since our market is assumed a complete market.

Proposition 4.1. Let $X_1, X_2, - - -, X_n$ be n independent standard normal random variables and let $\lambda \in \mathbb{R}$.Let the probability measure P^{λ} equivalent to P with density

$$\frac{P^{\lambda}}{P} = e^{(\lambda X_i - \frac{1}{2}\lambda^2)},$$

then the random Gaussian variables $X_1 - \lambda, X_2, , -, -, -X_n$ are independent standard variables with respect to P^{λ}.

Proof. We have to show that $X_1 - \lambda, X_2, -, -, -X_n$ are independent normally distributed random variables with respect to the probability measure P^{λ}. Since a random variable X with mean $E[X] = m$ and variance $E[X - E(X)]^2 = c$ is normally distributed if it characteristics function $E[e^{itX}] = e^{-\frac{1}{2}ct^2 + itm}$. Then

$$E_{P(\lambda)}[e^{itX_1}] = E_P[e^{itX_1}.e^{\lambda X_1} - \frac{1}{2}\lambda^2] \tag{5}$$

$$= E_P[e^{it+\lambda)X_1 - \frac{1}{2}\lambda^2}] \tag{6}$$

$$= e^{\frac{1}{2}(it+\lambda)^2 - \frac{1}{2}\lambda^2} \tag{7}$$

$$= e^{-\frac{1}{2}t^2 + it\lambda}. \tag{8}$$

Therefore X_1 is normal with mean λ and variance 1 with respect to P^{λ}.It follows that $X_1-\lambda$ is normal with mean zero and variance 1 with respect to $P^{(\lambda)}$. We show also that X_2 is normal with mean,

$$E_{P^{(\lambda)}}[X_2]=E_P[e^{\lambda X_1-\frac{1}{2}\lambda^2}X_2]=E_P[e^{\lambda X_1-\frac{1}{2}\lambda^2}]E_P(X_2)=0$$

and

$$E_{P(\lambda)}[X_2^2]=E_P[e^(\lambda X_1-\frac{1}{2}\lambda^2)X_2^2]=1$$

has variance 1. It shows that $X_i, i=1,2,---n$ i.e, $X_1,-,-,X_n$ are normally distributed with mean zero and variance 1.

To prove that $X_1-\lambda, X_2,,-,-,-X_n$ are uncorrelated, i.e.,

$$E_{P^{(\lambda)}}[(X_1-\lambda)X_j]=0, j=2,3,-,-,-n$$

$$E_{P^{(\lambda)}}[(X_1-\lambda)X_j]=E_P[(X_1-\lambda)X_j.e^{\lambda X_1-\frac{1}{2}\lambda^2}]=E_P[X_j]E[(X_1-\lambda)e^{\lambda X_1-\frac{1}{2}\lambda^2}]=0.$$

Corollary 4.2. Let $X_1,X_2,-,-,-X_n$ be an n-independent standard normal variables and $\lambda_1,\lambda_2,-,-\lambda_n\in\mathbb{R}$. Then

$$E_P[(S_i-S_j)^+]=e^{y_i+\frac{1}{2}\lambda_i^2}\Phi(\frac{y_i-y_j+\lambda_i^2}{\sqrt{\lambda}_i^2+\lambda_j^2})-e^{y_j+\frac{1}{2}\lambda_j^2}\Phi(\frac{y_i-y_j+\lambda_j^2}{\sqrt{\lambda}_i^2+\lambda_j^2})\quad i\neq j,$$

where

$$S_i=e^{\lambda_i X_i+y_i}$$

and

$$S_j=e^{\lambda_j X_j+y_j},\quad i,j=1,2,-,-,n.$$

Proof.

$$E_P[(S_i-S_j)^+]=E_P[(S_i-S_j)]$$

$S_i\geqslant S_j$

$$=E_P[e^{\lambda_i X_i+y_i}1_{\lambda_i X_i+y_i\geqslant\lambda_j X_j+y_j}]-E_P[e^{\lambda_j X_j+y_j}1_{\lambda_i X_i+y_i\geqslant\lambda_j X_j+y_j}]$$

$$=E_P[e^{\lambda_i X_i+y_i+\frac{1}{2}\lambda_i^2-\frac{1}{2}\lambda_i^2}]-E_P[e^{\lambda_j X_j+y_j+\frac{1}{2}\lambda_j^2-\frac{1}{2}\lambda_j^2}]$$

$$=e^{\frac{1}{2}\lambda_i^2+y_i}E_P[e^{\lambda_i X_i-\frac{1}{2}\lambda_i^2}]-e^{\frac{1}{2}\lambda_j^2+y_j}E_P[e^{\lambda_j X_j-\frac{1}{2}\lambda_j^2}],$$

By the above proposition, one has

$$= e^{\frac{1}{2}\lambda_i^2 + y_i} E_{P(\lambda)}[\lambda_i X_i + y_i \geqslant \lambda_j X_j + y_j] - e^{\frac{1}{2}\lambda_j^2 + y_j} E_{P(\lambda)}[\lambda_i X_i + y_i \geqslant \lambda_j X_j + y_j]$$

$$= e^{\frac{1}{2}\lambda_i^2 + y_i} P^{(\lambda_i)}[\lambda_i X_i + y_i \geqslant \lambda_j X_j + y_j] - e^{\frac{1}{2}\lambda_j^2 + y_j} P^{(\lambda_j)}[\lambda_i X_i + y_i \geqslant \lambda_j X_j + y_j]$$

(9)

$$= e^{\frac{1}{2}\lambda_i^2 + y_i} P^{(\lambda_i)}[\lambda_i (X_i - \lambda_i) - \lambda_j X_j \geqslant y_j - y_i - \lambda_i^2] - e^{\frac{1}{2}\lambda_j^2 + y_j} P^{(\lambda_j)}[\lambda_j (X_j - \lambda_j) - \lambda_i X_i \leqslant y_i - y_j - \lambda_j^2].$$

We have shown that the random variables $X_1 - \lambda_1, X_2, -, -, X_n$ are standard normal distribution with respect to $P^{(\lambda_i)}$, so that with respect to the same probability measure, the random variable $Z_i = \lambda_i(X_i - \lambda_i) - \lambda_j X_j$, $j \neq i$ has a normal distribution with mean zero(vector) and co-variance matrix $\sigma_i^2 = \lambda_{ik}^2 + \lambda_{jk}^2 \; i > j$. But $\sigma_i^2 \quad \forall i$, are equal, so that

$$\sigma_i^2 = \sigma_{i+1}^2 = -- = \sigma_{i+(n-1)}^2,$$

Then (9) becomes

$$e^{y_i + \frac{1}{2}\lambda_i^2} \Phi(\frac{y_i - y_j + \lambda_i^2}{\sqrt{\lambda_i^2 + \lambda_j^2}}) - e^{y_j + \frac{1}{2}\lambda_j^2} \Phi(\frac{y_i - y_j + \lambda_j^2}{\sqrt{\lambda_i^2 + \lambda_j^2}}).$$

If we consider a portfolio with 3 underlying assets i.e., $n = 3$ then the above has the following expression

$$e^{y_1 + \frac{1}{2}\lambda_1^2} \Phi(\frac{y_1 - y_2 + \lambda_1^2}{\sqrt{\lambda_1^2 + \lambda_2^2}}) - e^{y_2 + \frac{1}{2}\lambda_2^2} \Phi(\frac{y_1 - y_2 + \lambda_2^2}{\sqrt{\lambda_1^2 + \lambda_2^2}}) \tag{10}$$

$$e^{1 + \frac{1}{2}\lambda_1^2} \Phi(\frac{y_1 - y_3 + \lambda_1^2}{\sqrt{\lambda_1^2 + \lambda_3^2}}) - e^{y_3 + \frac{1}{2}\lambda_3^2} \Phi(\frac{y_1 - y_3 + \lambda_3^2}{\sqrt{\lambda_1^2 + \lambda_3^2}}) \tag{11}$$

$$e^{y_2 + \frac{1}{2}\lambda_2^2} \Phi(\frac{y_2 - y_3 + \lambda_2^2}{\sqrt{\lambda_2^2 + \lambda_3^2}}) - e^{y_3 + \frac{1}{2}\lambda_3^2} \Phi(\frac{y_2 - y_3 + \lambda_3^2}{\sqrt{\lambda_2^2 + \lambda_3^2}}). \tag{12}$$

5. Pricing and hedging Portfolio of best of assets option

For a fixed time interval $[0,T]$, the random variables $X_i, i=1,2,-,-,n$ are Brownian motion $B_i(T,w)$ respectively.The equivalent probability measure $P^{(\overrightarrow{u})}$ will be given by the density

$$dP^{(\overrightarrow{u})}(w) = e^{\overrightarrow{u}X_i - \frac{T}{2}||\overrightarrow{u}||^2},$$

where we have assume that the vector $\overrightarrow{u}$ satisfies the Novikov condition.

Proposition 5.1. The price of the European Best of Asset option is given by

$$V(0) = X_i(0)\Phi(\frac{ln(\frac{X_i(0)}{K}) + \frac{T}{2}\sum_{j=1}^{n} C_{ij}}{\sqrt{T\sum_{j=1}^{n} C_{ij}}}),$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}Z^2} dz$ is the cumulative distribution function of the standard normal distribution.

Proof. With respect to the EMM Q,the price of the n underlying assets $X_i, i=1,2,-,-n$ are given by

$$X_i(t) = X_i(0)e^{(\rho - \frac{1}{2} + \sum_{j=1}^{n} C_{ij})t + \sum_{j=1}^{n} C_{ij}B_j(t)} = X_i(0)e^{\rho t}.e^{-\frac{T}{2}||\overrightarrow{u}_i||^2 + \overrightarrow{u}_i B_i(t)}, i=1,2,-,-n,$$

where

$$\overrightarrow{u}_i = (\sigma_{i1}, \sigma_{i2}, -, -, \sigma_{in})^{Tr},$$

$B_i(t) = (B_1(t), B_2(t), -, -, B_n(t))^{Tr}$. Therefore,the time zero price of the European Best of Asset option is

$$V(0) = E_Q[e^{-\rho T} f(w)] = E_Q[X_i(T) - K]$$

$$= E_Q[X_i(0)e^{-\frac{T}{2}||\overrightarrow{u}_i||^2 + \overrightarrow{u}_i B_i(T)} - K].$$

If we define the probability measure $Q^{(u_i)}, i=1,2,-,-n$ equivalent to Q by

$$dQ^{(u_i)}(w) = e^{-\frac{T}{2}||\overrightarrow{u}_i||^2 + \overrightarrow{u}_i B_i(T)},$$

then

$$E_Q[X_i(0)e^{-\frac{T}{2}||\overrightarrow{u}_i||^2 + \overrightarrow{u}_i B_i(T)} - K]$$

$$= X_i(0)E_Q(u_i)[1_{\overrightarrow{u}_i B_i(T) - K \geq ln(\frac{X_i(0)}{K}) + \frac{T}{2}||\overrightarrow{u}_i||^2}].$$

Considering the result of the previous proposition,we then state that the price at time zero of the European call for Best of Asset option is given by $V(0) = X_i(0)N(d_1)$ where

$$d_1 = \frac{In(\frac{X_i(0)}{K}) + \frac{T}{2}||u_i||^2}{\sqrt{T||u_i||}^2}.$$

Also,the time zero price of the European put for Best of Asset option is given by $V(0) = X_i(0)N(d_2)$, where

$$d_2 = \frac{In(\frac{X_i(0)}{K}) - \frac{T}{2}||u_i||^2}{\sqrt{T||u_i||}^2}$$

$||u_i||^2 = \sum_{j=1}^{n} \sigma_{i,j}^2$. It is important to observe that this price depend only on the market volatilities.

We consider the portfolio

$$\Theta = (\theta_0(t), \theta_1(t), \theta_2(t), ...\theta_n(t)).$$

Using the CHO formula, we have

$$\Gamma(t) = e^{-\rho(T-t)}\sigma^{-1}E_Q[D_tF/\mathscr{F}_t],$$

where σ^{-1} = inverse of the matrix nxn matrices, and $\Gamma(t) = (\theta_1(t), \theta_2(t), ...\theta_n(t))$ $D_tF = (\sigma_{i1}, \sigma_{i2}, ..., \sigma_{in})^{Tr}X_i(T)1_D - K)$, where $D = w : X_i(T,w) > K$, therefore, we have

$$E_Q[D_tF/\mathscr{F}_t] = (\sigma_{i1}, \sigma_{i2}, ..., \sigma_{in})^{Tr}E_Q[X_i(T)1_D\ \mathscr{F}_t - K).$$

Proposition 5.2. The perfect Hedge $\Theta(t)$ of the option is given by

$$\Theta_i(t) = \frac{1}{\Delta}[X_i(t)(\sigma_{i,i}\sigma_{j,j} - \sigma_{i,j}^2)\Phi(d_i) - X_j(t)(\sigma_{j,i}\sigma_{j,i} - \sigma_{i,j}\sigma_{j,j})\Phi(d_j)].$$

Proof. To compute $E_Q[X_i(T)1_D/\mathscr{F}_t]$ $\quad i = 1, 2, ...n$,we need the markov property.
We compute $E_Q[X_i(T)1_D/\mathscr{F}_t]$ as follow:

$$E_Q[X_i(T)1_D/\mathscr{F}_t] = E_Q[X_i(T)1_{X_i(T)\geq K}/\mathscr{F}_t]$$

$$= E_Q[F(X_i(T))/\mathscr{F}_t] = E_Q^{X_i,K}[F(Y(T-t)/\mathscr{F}_t],$$

where $f(y) = f(X_i, K) = 1_{X_i \geq K}$. Therefore, the previous expression becomes $E_Q^{X_i,K}[X_i(T-t)1_{X_i(T-t)\geq K}]$. But with respect to Q, we have

$$X_i(t) = X_i(0)e^{(\rho - \frac{1}{2}\sum_{j=1}^n \sigma_{i,j}{}^2)t} + \sum_{j=1}^n \sigma_{i,j}B_j(t), i = 1,2...n.$$

Since $Y(T-t)$ is independent of $\mathscr{F}_t$, we have

$$E_Q^{X_i,K}[F(Y(T-t)\ \mathscr{F}_t]_{X_i = X_i(t)}$$

$$= E_Q[X_i(t)e^{(\rho - \frac{1}{2}\sum_{j=1}^n \sigma_{i,j}^2)(T-t)} + \sum_{j=1}^n \sigma_{i,j}B_j(T-t)1_D]$$

$$= X_i(t)e^{\rho(T-t)}E_Q[e^{(-\frac{T-t}{2}\sum_{j=1}^n \sigma_{i,j}^2)t} + \sum_{j=1}^n \sigma_{i,j}B_j(T-t)1_D],$$

where

$$D = \sum_{j=1}^n \sigma_{1,j}B_j(T-t) - \sum_{j=1}^n \sigma_{2,j}B_j(T-t) \geq \frac{T-t}{2}\sum_{j=1}^n (\sigma_{1,j}^2 - \sigma_{2,j}^2) + In(\frac{X_i}{K}).$$

Since $\sum_{j=1}^n (\sigma_{1,j}^2) = ||U_i||^2$, we have $= X_i(t)e^{\rho(T-t)}E_Q[e^{(-\frac{T-t}{2}||U_i||^2 + U_iB(T-t)}1_D]$ so that

$$D = \overrightarrow{U}_1\overrightarrow{B}(T-t) - \overrightarrow{U}_2\overrightarrow{B}(T-t) \geq \frac{T-t}{2}||\overrightarrow{U}_2|| + In(\frac{X_1(t)}{K})$$

$\overrightarrow{B}(T-t) = \left\| B_1(T-t) \quad B_2(T-t) \quad ,\dots \quad ,B_n(T-t) \right\|$ This is normally distributed with mean zero and Variance $(T-t)$ with respect to the measure Q. Then,$Z(T-t) = \frac{B(T-t)-0}{\sqrt{(T-t)}}$ is a normally distributed random vector with zero vector mean and covariance identity matrix.Then our expression become

$$X_i(t)e^{\rho(T-t)}E_Q[e^{(-\frac{T-t}{2}||U_i||^2 + U_i\sqrt{(T-t)}Z(T-t)}1_D]$$

so that

$$D = \overrightarrow{U}_1\sqrt{T-t}Z_1(T-t) - \overrightarrow{U}_2\sqrt{T-t}Z_2(T-t) \geq \frac{T-t}{2}||U_i||^2 + \frac{T-t}{2}||U_2||^2 + In(\frac{X_i(t)}{K})$$

Re-arranging our expression, we have that

$$X_i(t)e^{\rho(T-t)}E_Q[e^{-(\frac{T-t}{2})}||\overrightarrow{U}_i||^2 + \overrightarrow{U}_i\sqrt{(T-t)}Z(T-t)1_D]$$

$$D = \sqrt{T-t}\overrightarrow{U}_iZ(T-t) \geq \frac{T-t}{2}||\overrightarrow{U}_i||^2 + In(\frac{X_i(t)}{K})$$

so that $= X_i(t)e^{\rho(T-t))}\Phi(d)$

$$d = \frac{In(\frac{X_i(t)}{K}) + \frac{T-t}{2}||\overrightarrow{U_i}||^2}{\sqrt{T-t}||U_i||^2}$$

$$E_Q[D_tF/\mathscr{F}_t] = (\sigma_{i1}, \sigma_{i2}, ..., \sigma_{in})^T X_i(t)e^{\rho(T-t))}\Phi(d_i) - (\sigma_{j1}, \sigma_{j2}, ..., \sigma_{jn})^T X_j(t)e^{\rho(T-t))}\Phi(d_j)$$

$$\Gamma(t) = e^{-\rho(T-t)}\sigma^{-1}E_Q[D_tF/\mathscr{F}_t]$$

gives

$$\Gamma(t) = (\sigma_{i1}, \sigma_{i2}, ..., \sigma_{in})^T X_i(t)\sigma^{-1}\Phi(d_i) - (\sigma_{j1}, \sigma_{j2}, ..., \sigma_{jn})^T X_j(t)\sigma^{-1}\Phi(d_j).$$

Conclusion. In this paper, we have developed an explicit formulation with the generalized Clark-Haussmann -Ocone CHO formula for the price and hedging of portfolio of n-risky assets.This has a direct implication on European contingent claim.This work can be extended if we considered a stratonovich approach for the equations (3) and (4) and investigate the possibility of having a solution in relation to the malliavin calculus developed on CHO.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Pascucci, PDE and Martingale Methods in Option Pricing, Bocconi and Springer Series, 2010.

[2] C. Alexander, A. Venkatramanan, Analytic Approximation for Spread Options.International Capital Market Association(ICMA), 2009.

[3] G.O.S. Ekhaguere, An Invitation to the pricing of contingent claims, International Centre for Mathemtical and Computer Science, 1 (2004), 197-214.

[4] K. Aase, B. Oksendal, N. Privault, J. Uboe, White Noise Generalization of the Clack -Haussmann -Ocone Theorem with application to Mathematical Finance, Finance and Stochastic, 4 (2000), 465-496.

[5] M. William, The Value of an option to exchange one asset for another, J. Finance 33 (1978), 177-186.

[6] R. Carmona, V. Durrleman, Pricing and hedging spread options, SIAM Rev. 45 (2005), 627-685.

[7] S. Mataramvura, The Malliavin Derivative and Application to Pricing and Hedging a European exchange Option, J. Math. Finance, 2 (2012), 280-290.

[8] Z. Yang, C.O. Ewald, O. Menken, Pricing and Hedging of Asian Option: Quasi-Explicit Solution via Malliavin Calculus, 2009.