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AN ELEMENTARY LINEAR FUNCTIONAL APPROACH TO THE FUNDAMENTAL THEOREM OF ASSET PRICING

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Abstract: An elementary linear functional analytic proof of the fundamental theorem of asset pricing for the one-period model of the financial market with a finite state space is proposed. The introductory presentation is largely inspired from Pliska. It is based on a special case of the separating hyper-plane theorem. This equivalent form of the classical Hahn-Banach separation theorem is proved by means of standard tools from linear algebra, vector geometry and convex analysis.

Keywords: asset pricing; arbitrage; risk-neutral valuation; linear algebra; convex analysis; Hahn-Banach separation theorem; separating hyper-plane theorem.

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1. Introduction

One of the most important accomplishments of the 20^{th} century is the development of modern mathematical finance, which is of tremendous significance for both theory and applications. Besides the early formulation of portfolio theory through Markowitz [22], [23], and the formula

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by Black and Scholes [2] for the valuation of options, the third foundation pillar is the fundamental theorem of asset pricing, which goes back to pioneer works by Harrison and Kreps [14], and Harrison and Pliska [15].

Starting point is the notion of arbitrage, which represents a risk-free possibility to acquire money. From the economic viewpoint no arbitrage strategies should exist in financial markets. According to the fundamental theorem of asset pricing, this is the case if there is a risk-neutral probability measure on the state space of possible events.

Although all mathematical finance textbooks contain a treatment of this primordial result (e.g. Duffie [9], Pliska [24], Karatzas und Shreve [18], Shiryaev [27], Elliot and Kopp [10], and Föllmer and Schied [11]), all derivations, up to Pliska [24], rely on prerequisites, which go far beyond the prevalent mathematical curriculum vitae. Therefore, it is first priority to offer a larger audience of interested readers an elementary approach to this fundamental theorem.

Our introductory presentation is inspired from [24] and is based on a special case of the classical Hahn-Banach separation theorem, or equivalently the separating hyper-plane theorem in the Euclidean space. Section 2 derives this result with elementary means from linear algebra, vector geometry and convex analysis. Section 3 introduces the essential one-period model of the financial market with a finite state space and the notion of arbitrage strategy. Finally, the fundamental theorem of asset pricing for the one-period model is formulated and proved in Section 4.

2. The Hahn-Banach separation theorem for a linear subspace

Separation theorems for convex sets, as first derived by Hahn [13] and Banach [1], are significant mathematical resources (see e.g. van Tiel [28], Klee [19] and their references). They find an enormous and widespread application in economics and finance mathematics, from the theory of games (e.g. Blackwell and Girshick [3], Chapter 2, Kuhn [20], Chapter 2) to the fundamental theorem of mathematical finance (e.g. Pliska [24], Chapters 1 and 3, Föllmer and Schied [11], Theorems 1.6 and 5.17).

The separating hyper-plane theorem is to such an extent important that besides abstract proofs (e.g. Gale [12], p.44) and geometric derivations (e.g. Debreu [6]) even an economics proof by Weitzmann [29] has been proposed. The formulation of separation theorems in the language of linear algebra, geometry and convex analysis requires the following mathematical prerequisites.

A subset $C \subset R^n$ is called *convex* if given two points $x, y \in C,$ the connecting line between these two points also belongs to the subset, i.e. $\lambda x + (1-\lambda) y \in C$ for all $\lambda \in [0,1]$. The subset $C \subset R^n$ is called *compact* if it is *closed* (it contains all its limit points) and *bounded* (all its points lie within some fixed distance from each other). A subset $L \subset R^n$ is a *linear subspace* if it is a real *vector space*. A map $f : R^n \to R$ is a *linear map* if it satisfies the condition $f(\lambda x + \nu y) = \lambda f(x) + \nu f(y)$ for all $x, y \in R^n$, $\lambda, \nu \in R$. The symbol $\circ$ denotes the *scalar product* in R^n, also called *dot product* or *inner product*. For two vectors $x = (x_1,..., x_n)$, $y = (y_1,..., y_n) \in R^n$, it is defined by $x \circ y = \sum_{i=1}^{n} x_i y_i$. A *hyper-plane* $H \subset R^n$ through a point $a \in R^n$ is the translation of a linear subspace in R^n, which does not go through the origin, and has the set representation $H = \{ x \in R^n \mid x \circ a = d \},\ d \in R - \{0\}$. A hyper-plane divides the Euclidean space in two subspaces $H^+, H^- \subset R^n$, which are defined by $H^+ = \{ x \in R^n \mid x \circ a \geq d \}$, $H^- = \{ x \in R^n \mid x \circ a \leq d \}$. The hyper-plane $H \subset R^n$ is said to *separate* two subsets $A, B \subset R^n$ if the latter lie in two different half-spaces, i.e. $A \subset H^+, B \subset H^-$ or $A \subset H^-, B \subset H^+$. The hyper-plane *strictly separates* the subsets if it separates them with an empty intersection, i.e. $H \cap A = \emptyset, H \cap B = \emptyset$.

We are ready for the formulation of the Hahn-Banach separation theorem.

Theorem 2.1. *Let* $C \subset R^n$ *be a compact convex set with* $0 \notin C$, *and let* $K \subset R^n$ *be a closed convex set. If* $C \cap K = \emptyset$, *then there exists a non-zero linear map* $f : R^n \to R$ *that satisfies* $\sup_{x \in K} f(x) < \inf_{y \in C} f(y)$.

An equivalent geometric formulation is obtained by means of the famous theorem of Riesz-Fisher for the n-dimensional Euclidean space.

Theorem 2.2. *Let* $L \subset R^n$ *a linear subspace and* $f : R^n \to R$ *a linear map. Then there exists a uniquely determined point* $a \in R^n$ *such that* $f(x) = x \circ a$ *for all* $x \in L$.

Proof. One shows first the existence of $a \in R^n$. If $f(x) = 0$ one puts $a = 0$. Suppose now that $f(x) \neq 0$. The image of the linear map defined by $\operatorname{Im} f = \{ y \in R \mid y = f(x), x \in R^n \}$ has dimension one, i.e. $\dim(\operatorname{Im} f) = 1$. Denote the kernel of the linear map by $Ker f = \{ x \in R^n \mid f(x) = 0 \}$. Linear algebra implies the relationship

$$\dim L = \dim(Ker f) + \dim(\operatorname{Im} f) = \dim(Ker f) + \dim(Ker f)^{\perp},$$

where $(\cdot)^{\perp}$ denotes the orthogonal complement. Since $\dim(Ker\ f)^{\perp} = 1$ there exists $e \in L$, which is a basis vector of $(Ker\ f)^{\perp}$. An arbitrary $x \in L$ can be uniquely decomposed as $x = x' + \lambda e,\ x' \in Ker f,\ \lambda \in R$. Since e is orthogonal to x', one has

$$x \circ e = x' \circ e + \lambda \cdot e \circ e = \lambda \cdot e \circ e,$$

which implies that $\lambda = \frac{x \circ e}{e \circ e}$. Since f is a linear map one obtains for all $x \in L$ that $f(x) = f(x' + \lambda e) = f(x') + \lambda f(e) = \lambda f(e)$. It follows that

$$f(x) = \frac{x \circ e}{e \circ e} \cdot f(e) = x \circ a, \quad \text{with} \quad a = \frac{f(e) \cdot e}{e \circ e}.$$

Uniqueness of a is immediate. Indeed, if a and a' are such that $f(x) = x \circ a = x \circ a'$, then one has $x \circ (a' - a) = 0$ for all $x \in L$, which is only possible if $a' = a$. ◊

Applied to the Hahn-Banach separation theorem, there exists $a \in R^n$ such that $\sup_{x \in K}\{x \circ a\} < \inf_{y \in C}\{y \circ a\}$. But the equation $x \circ a = c$ defines a hyper-plane in R^n. This leads to the following geometric equivalent form of Theorem 2.1.

Theorem 2.3. (Separating hyper-plane theorem). *Given are two convex sets* $C, K \subset R^n$ *such that* $C \cap K = \emptyset$. *Assume that* $0 \notin C$, C *is compact and* K *is closed. Then there exists a hyper-plane* $H \subset R^n$ *that strictly separates* C *and* K.

Proof. We follow Blackwell and Girshick [3], pp.34-35. Consider the shortest distance between C and K in the Euclidean norm $\|x\| = \sqrt{x \circ x}$, $x \in R^n$, which is given by $d(C,K) = \inf_{x \in C, y \in K} \|x - y\|$. Since C is closed and bounded, and K is closed, there exist sequences $\{x_n\}, \{y_n\}$, with $x_n \in C$, $y_n \in K$ such that $\lim_{n \to \infty} \|x_n - y_n\| = d(C,K)$ and $\{y_n\}$ is bounded. This follows from the inequality $\left\| |x_n| - |y_n| \right\| \le \|x_n - y_n\|$ and the fact that $\{x_n\}$ is bounded. The theorem of Bolzano-Weierstrass (e.g. Blatter [4], (6.22), p.82) implies the existence of points $x_0 \in C$ and $y_0 \in K$, for which the shortest distance between C and K is attained, i.e. one has $\|x_0 - y_0\| = \min_{x\in C, y\in K} \|x - y\| = d(C,K)$. We claim that the hyper-plane H, which goes through the middle point $z_0 = \frac{1}{2}(x_0 + y_0)$ of the line between x_0 and y_0, and is normal to this line, that is $H = \{ z \in R^n \mid (x_0 - y_0) \circ z = c, c = (x_0 - y_0) \circ z_0 \}$, strictly separates the sets C and K. To show this, consider an arbitrary point z with $(x_0 - y_0) \circ z \le c$. Then, the square of the distance between y_0 and a point of the line between x_0 and z is given by

$$\phi(\lambda) = \|x_0 + \lambda(z - x_0) - y_0)\|^2 = \|z - x_0\|^2 \lambda^2 + 2(z - x_0) \circ (x_0 - y_0)\lambda + \|x_0 - y_0\|^2.$$

The derivative of this quadratic function at $\lambda = 0$ yields the inequality

$$\phi'(0) = 2(z - x_0) \circ (x_0 - y_0) = 2(x_0 - y_0) \circ z - 2(x_0 - y_0) \circ x_0 \le 2c - 2(x_0 - y_0) \circ x_0 .$$

By definition of the constant c one has

$$(x_0 - y_0) \circ x_0 - c = (x_0 - y_0) \circ (x_0 - \tfrac{1}{2}(x_0 + y_0)) = \tfrac{1}{2}\|x_0 - y_0\|^2 > 0 ,$$

which implies that $\phi'(0) < 0$. Therefore, there exist points w on the line between x_0 and z with $\|w - y_0\| < \phi(0) = \|x_0 - y_0\| = d(C,K)$, which implies the condition $z \notin C$ by the convexity property. Since z is an arbitrary point with $(x_0 - y_0) \circ z \le c$, one has $(x_0 - y_0) \circ x > c$ for all $x \in C$, hence $C \subset H^+$ and $C \cap H = \varnothing$. In the same way, one shows that $K \subset H^-$ and $K \cap H = \varnothing$. Together this shows that the hyper-plane strictly separates the sets C and K. ◊

For the main application in Section 4, one needs the following specialization of the Hahn-Banach separation theorem for a linear subspace.

Corollary 2.1. *Given are two convex sets* $C, K \subset R^n$ *such that* $C \cap K = \varnothing$. *Assume that* $0 \notin C$, C *is compact and* K *is a linear subspace. Then, there exists a linear map* $f : R^n \to R$ *with* $f(y) > 0$ *for all* $y \in C$, *and* $f(x) = 0$ *for all* $x \in K$.

Proof. One notes that the assumptions of Theorem 2.1 are fulfilled because a linear subspace is closed. This theorem implies that $\sup_{x \in K} f(x) < \inf_{y \in C} f(y)$. Since K is a linear subspace, one has $0 \in K$ and $\sup_{x \in K} f(x) \geq f(0) = 0$. If the supremum is strictly positive, then the linearity of the map f enforces that $f(x)$ is unbounded (use that $f(cx) = cf(x)$ for all $c > 0$). But this is a contradiction to the statement that the supremum is strictly smaller than $\inf_{y \in C} f(y)$. Even more, one must have $f(x) = 0$ for all $x \in K$. Indeed, if $f(x) < 0$ for some $x \in K$, then by linearity one has $f(-x) = -f(x) > 0$, in contradiction to $\sup_{x \in K} f(x) = 0$. This shows that Corollary 2.1 follows from Theorem 2.1 or the equivalent Theorem 2.2. ◊

Remark 2.1. In Theorem 2.3 (or Theorem 2.1) the assumption that C is compact, is necessary. As a counterexample the convex and closed sets in R^2, which are given by

$$C = \{ (x, y) \in R^2 \mid x > 0,\ y \leq -\tfrac{1}{x} \} \quad \text{and} \quad K = \{ (x, y) \in R^2 \mid x > 0,\ y \geq \tfrac{1}{x} \},$$

cannot be separated by a hyper-plane. Among all possible lines, only the line $y = 0$ separates the sets C and K. This is a one-dimensional subspace of R^2 but not a hyper-plane, which by definition does not meet the zero-point $(0,0)$.

3. One-period model of the financial market with a finite state space

The one-period model of the financial market is specified by the following assumptions:

(A1) There is a starting date $t=0$ and a terminal date $t=1$, and financial transactions at these dates are allowed.

(A2) There is a finite state space $\Omega=\{\omega_1,...,\omega_n\}$, where each event $\omega\in\Omega$ represents a possible state of the world, which is unknown at time $t=0$, and disclosed to the investor at time $t=1$.

(A3) There is probability measure P on Ω with $P(\omega)>0$ for all $\omega\in\Omega$.

(A4) There is a risk-free interest rate $r\geq 0$. If a unit of money is invested at time $t=0$ in the risk-free asset, then its value at time $t=1$ is the amount $1+r$.

(A5) There are m risky assets, whose prices at time $t\in\{0,1\}$ are determined by the price vector $S(t)=\left(S_1(t),...,S_m(t)\right)$. At the starting date the prices $S_k(0)$ are known positive constants, but the prices $S_k(1)$ are non-negative random variables, whose values are known to the investor only at terminal date $t=1$.

To describe the financial market activities, one needs the following important notions. A trading strategy $H=\{H_0,H_1,...,H_m\}$ represents the portfolio an investor holds in the time interval $[0,1]$. The quantity H_0 is the amount invested in the risk-free asset while H_k, $k\in\{1,...,m\}$, is the number of shares, which are invested in the k-th risky asset. The discounted value process of the portfolio represents the value of the whole investment at the times $t=0,1$, and is defined by

$$V_0=H_0+\sum_{k=1}^{m}H_kS_k(0),\quad V_1=H_0+\frac{1}{1+r}\cdot\sum_{k=1}^{m}H_kS_k(1). \tag{3.1}$$

On recognizes immediately that the discounted profit of the portfolio is determined by the discounted changes in value of the risky assets given by $\Delta S_k = \frac{1}{1+r} S_k(1) - S_k(0)$, $k \in \{1,..., m\}$. A formula for it is

$$G = V_1 - V_0 = \sum_{k=1}^{m} H_k \cdot \Delta S_k \ . \tag{3.2}$$

An arbitrage strategy is a special trading strategy that yields a strictly positive expected profit, and is characterized by one of the following equivalent conditions:

$$V_0 = 0, V_1 \geq 0 \quad \text{and} \quad E^P[V_1] > 0, \tag{3.3}$$

$$G \geq 0 \quad \text{and} \quad E^P[G] > 0 \, . \tag{3.4}$$

To show the equivalence of these conditions, suppose that H is an arbitrage strategy. From (3.2) one has $G = V_1 - V_0$, hence (3.4) follows from (3.3). Conversely, suppose that (3.4) holds for a trading strategy $\hat{H}$. Consider the trading strategy $H = \{H_0, \hat{H}_1,..., \hat{H}_m\}$, with

$$H_0 = -\sum_{k=1}^{m} \hat{H}_k S_k(0) \, . \tag{3.5}$$

For this trading strategy one has $V_0 = 0$. Further, one has $V_1 = V_0 + G = G$, which implies that (3.4) implies (3.3), and H is an arbitrage strategy. Translated in everyday language, an arbitrage strategy is a risk-free possibility to earn money. One starts with nothing and, without any danger to get into debts, there is a chance to get a positive amount of money. It is therefore easy to understand that from the economics viewpoint arbitrage strategies should not exist in financial markets. A necessary and sufficient condition for this is provided by the very important fundamental theorem of asset pricing. The next Section presents and proves this result for the considered one-period model of the financial market.

4. Fundamental theorem of asset pricing

The pursued path below is the mathematical quintessence of the approach in [24], Chapter 1.3. The same result for a multi-period of the financial market with a finite state space and a finite time horizon is also found in [24], Chapter 3.4. These results go back to Harrison and Kreps [14], and Harrison and Pliska [15]. It is remarkable that the fundamental theorem of asset pricing is valid for more general models of the financial market. These generalizations admit infinite state spaces or/and an infinite time horizon. However, the derivation of the extended results need advanced mathematical tools, which go far beyond the scope of an elementary presentation. An introduction to this issue is found in [24], Chapter 7. For alternative treatments of the fundamental theorem we refer to Kabanov and Kramkov [16], Kabanov and Stricker [17], Föllmer and Schied [11], as well as to the advanced and specialized developments in Dalang, Morton and Willinger [5], Schachermayer [25], [26], and Delbaen and Schachermayer [7], [8].

To formulate the fundamental theorem one needs the following important notion. A risk-neutral probability measure Q on Ω is a probability measure with the properties

$$Q(\omega) > 0 \quad \text{for all} \quad \omega \in \Omega, \tag{4.1}$$

$$\frac{1}{1+r} \cdot E^{Q}\left[S_k(1)\right] = S_k(0), \quad k \in \{1,..., m\}. \tag{4.2}$$

The equation (4.2) requires that the expected discounted price of any risky asset coincides with its starting price. The condition (4.1) tells us that any event of the state space occurs with a strictly positive probability. According to the assumption (A.3) this also holds for the underlying real-world probability measure P on Ω. One says that P and Q are equivalent probability measures. We derive now the simplest version of the fundamental theorem of asset pricing.

Theorem 4.1. *The one-period model of the financial market with a finite space is free of arbitrage if, and only if, there exists a risk-neutral probability measure.*

Proof. To enable the application of the results of Section 2, we identify the set of random variables on Ω with a Euclidean space as follows. A real random variable $X : \Omega \to R$ corresponds to a point $x = (x_1, ..., x_n) \in R^n$, with $x_k = X(\omega_k),\ k = 1, ..., m$. The derivation is carried out in two parts.

Part 1: The condition is sufficient

Let Q be a risk-neutral probability measure and H an arbitrary trading strategy with the property $V_1 \geq 0$ and $E^P[V_1] > 0$. With (3.1) and (4.2) one has

$$V_0 = H_0 + \sum_{k=1}^{m} H_k S_k(0) = H_0 + \frac{1}{1+r} \cdot \sum_{k=1}^{m} H_k E^Q[S_k(1)] = E^Q[V_1] > 0 .$$

Since H is arbitrary, one sees that (3.3) can never be fulfilled, which shows that there is no arbitrage strategy.

Part 2: The condition is necessary

It is shown that if the financial market is free of arbitrage, then there exists a risk-neutral probability measure. Consider the set of all probability measures

$$C = \{\, x \in R^n \;\Big|\; x_k \geq 0,\ \sum_{k=1}^{n} x_k = 1 \,\}, \tag{4.3}$$

and the set of all realizable discounted profit processes

$$K = \{\, x \in R^n \;\big|\; \exists\, H \in R^{n+1} \quad with \quad x = G = \sum_{k=1}^{m} H_k \Delta S_k \,\}. \tag{4.4}$$

The set C is convex, compact and $0 \notin C$, and the set K is a linear subspace, which is convex and closed. Further, consider the set of non-negative random variables on Ω, which one identifies with the subset $A = \{\, x \in R^n \mid x_k \geq 0,\ k = 1, ..., n \,\}$ of the Euclidean space. According to (3.4) the financial market is free of arbitrage if, and only if, one has $K \cap A = \{0\}$.

Now, one has $C \subset A$ and $0 \notin C$. The arbitrage free assumption implies that $C \cap K = \varnothing$. The sets C and K satisfy herewith the assumptions of Corollary 2.1. Therefore, there is a linear map $f : R^n \to R$ with $f(y) > 0$ for all $y \in C$, and $f(x) = 0$ for all $x \in K$. Since the points $e_k = (0,...,0,1,0,...,0)$ (one at the k-th place and zero elsewhere) belong to the set C, one has $f(e_k) > 0,\ k = 1,...,n$. It follows that

$$Q_k = Q(\omega_k) = \frac{f(e_k)}{\sum f(e_k)} > 0, \quad k = 1,...,m, \quad \sum Q_k = 1,$$

yields a probability measure on Ω. A random variable X of the discounted profit process, which is represented by a point $x \in K$, satisfies by Corollary 2.1 the property

$$E^Q[X] = \sum Q_k x_k = \frac{\sum f(e_k) x_k}{\sum f(e_k)} = \frac{f\left(\sum x_k e_k\right)}{\sum f(e_k)} = \frac{f(x)}{\sum f(e_k)} = 0. \tag{4.5}$$

In particular, for each $k = 1,...,m$ the random variable $X_k = \Delta S_k$ belongs to K. On the other hand, the equation (4.5), that is $E^Q[\Delta S_k] = 0$, is equivalent with the statement

$$\frac{1}{1+r} E^Q[S_k(1)] = S_k(0), \quad k = 1,...,m. \tag{4.6}$$

Finally, with (4.4) and (4.6) one sees that Q is a risk-neutral probability measure. ◊

Conflict of Interests

The author declares that there is no conflict of interests.

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