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THE SCHWARTZ AND SMITH (2000) MODEL WITH STATE-DEPENDENT RISK PREMIA

ALESSANDRO SBUELZ

Department of Mathematical Sciences, Mathematical Finance and Econometrics,

Catholic University of Milan, Milan, Italy

Abstract. In this paper, I prove the closed-form extension of the Schwartz and Smith (2000) model of commodity futures pricing to state-dependent risk premia. The extended model exhibits important additional flexibility in representing different term-structure patterns.

Keywords: Commodity futures pricing; Term structure of futures prices; State-dependent risk premia; Normal backwardation; Backwardation.

2010 AMS Subject Classification: 60G10, 60G15.

1. Introduction

The Schwartz and Smith (2000) model of commodity futures pricing has been widely used in the theoretical and empirical literature on commodity spot and derivatives markets, as it provides a way to disentangle the permanent ‘equilibrium’ component of the commodity spot price from its transitory component via futures price data. Primed by the studies of Fama

*Corresponding author
E-mail address: alessandro.sbuelz@unicatt.it
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and French (1987) and of Casassus and Collin-Dufresne (2005) on the importance of time-varying risk premia in commodity markets, Mirantes, Población, and Serna (2015) have recently proposed an important extension of the Schwartz and Smith (2000) model that considers state-dependent risk premia. Mirantes, Población, and Serna (2015) work out the general risk-neutral valuation scheme for a range of commodity contingent claims without, however, providing a fully explicit solution for the futures prices. Their main concern is investigating the impact of time-varying risk premia on commodity American options. I contribute (1) by deriving the fully closed form of futures prices from no-arbitrage restrictions written under the physical measure, which highlight the presence of the state-dependent risk premia, and (2) by detailing the incremental impact of such risk premia on the term structure of the futures prices.

2. No-arbitrage futures pricing

Schwartz and Smith (2000) assume that the spot log price of a given commodity is the sum of two components: $\ln(S_t) = \chi_t + \xi_t$. The non-stationary 'equilibrium' component ξ_t is an arithmetic Brownian motion with $\mathbb{P}$-dynamics $d\xi_t = \mu_\xi dt + \sigma_\xi dz_t^\xi$, where $\mathbb{P}$ is the physical probability measure. The stationary component χ_t is assumed to revert toward zero following an Ornstein-Uhlenbeck process with $\mathbb{P}$-dynamics $d\chi_t = -\kappa\chi_t dt + \sigma_\chi dz_t^\chi$ $(\kappa > 0)$. Under no arbitrage in the commodity derivatives markets, the state price density ζ_t has $\mathbb{P}$-dynamics

$$d\zeta_t = \zeta_t \left(-rdt - \Lambda_{\xi,t} dz_t^\xi - \Lambda_{\xi,t} dz_t^\chi\right),$$

where r is the riskfree rate. I depart from the Schwartz and Smith (2000) model by assuming state-dependent risk premia.

Assumption *The market prices of risk are state-dependent,*

$$\Lambda_{\xi,t} = \lambda_\xi + \phi_\xi \chi_t \quad \textit{(price of } \xi\textit{-type risk),}$$

$$\Lambda_{\chi,t} = \lambda_\chi + \phi_\chi \chi_t \quad \textit{(price of } \chi\textit{-type risk),}$$

and the speed of mean reversion remains positive after risk adjustment, $\kappa + \sigma_\chi \phi_\chi > 0$.

The original Schwartz and Smith (2000) model ensues by assuming away the dependence of $\Lambda_{\xi,t}$ and $\Lambda_{\chi,t}$ from the state χ_t ($\phi_\xi = 0$ and $\phi_\chi = 0$). Let $F(\xi_t, \chi_t, \tau)$ be the current futures price of the commodity for delivery in τ years. The no-arbitrage restriction under $\mathbb{P}$ for $F(\xi_t, \chi_t, \tau)$ emphasizes the presence of the state-dependent risk premia:

(1)
$$\begin{cases} E_t^{\mathbb{P}}[dF] &= \left(F_\xi \sigma_\xi \Lambda_{\xi,t} + F_\chi \sigma_\chi \Lambda_{\xi,t}\right) dt, \\ F(\xi_t, \chi_t, 0) &= \exp(\chi_t + \xi_t). \end{cases}$$

The resulting no-arbitrage futures price is characterized in the following proposition.

Proposition The function $F(\xi_t, \chi_t, \tau)$ that solves the problem (1) is

$$F(\xi_t, \chi_t, \tau) = \exp(\ \xi_t \ + \ \chi_t A(\tau) \ + \ B(\tau)\ ),$$

with

(2)
$$A(\tau) = \left(1 + \frac{\sigma_\xi \phi_\xi}{\kappa + \sigma_\chi \phi_\chi}\right) e^{-(\kappa + \sigma_\chi \phi_\chi)\tau} - \frac{\sigma_\xi \phi_\xi}{\kappa + \sigma_\chi \phi_\chi},$$

(3)
$$B(\tau) = D\tau + G\left(1 - e^{-2(\kappa + \sigma_\chi \phi_\chi)\tau}\right) + H\left(1 - e^{-(\kappa + \sigma_\chi \phi_\chi)\tau}\right)$$

$$\begin{aligned} D &= \left(\mu_\xi - \sigma_\xi \lambda_\xi + \frac{\sigma_\xi^2}{2}\right) - (\sigma_\xi \sigma_\chi \rho_{\xi\chi} - \sigma_\chi \lambda_\chi) \frac{\sigma_\xi \phi_\xi}{\kappa + \sigma_\chi \phi_\chi} + \frac{\sigma_\chi^2}{2} \frac{(\sigma_\xi \phi_\xi)^2}{(\kappa + \sigma_\chi \phi_\chi)^2}, \\ G &= \frac{\sigma_\chi^2 (\kappa + \sigma_\chi \phi_\chi + \sigma_\xi \phi_\xi)^2}{4(\kappa + \sigma_\chi \phi_\chi)^3}, \\ H &= \frac{(\kappa + \sigma_\chi \phi_\chi + \sigma_\xi \phi_\xi)\left[(\kappa + \sigma_\chi \phi_\chi)(\sigma_\xi \sigma_\chi \rho_{\xi\chi} - \sigma_\chi \lambda_\chi) - (\sigma_\xi \phi_\xi)\sigma_\chi^2\right]}{(\kappa + \sigma_\chi \phi_\chi)^3}. \end{aligned}$$

Proof. Under $\mathbb{P}$, the ex-ante marking-to-market instantaneous gain on being long the futures contract is

$$E_t^{\mathbb{P}}[dF] = \left(-F_\tau + F_\xi \mu_\xi - F_\chi \kappa \chi_t + \frac{1}{2} F_{\xi\xi} \sigma_\xi^2 + F_{\xi\chi} \sigma_\xi \sigma_\chi \rho_{\xi\chi} + \frac{1}{2} F_{\chi\chi} \sigma_\chi^2\right) dt,$$

where $d\left\langle z_t^{\xi}, z_t^{\chi}\right\rangle = \rho_{\xi\chi} dt$. Given the Ansatz $\exp(\xi_t + \chi_t A(\tau) + B(\tau))$, the no-arbitrage pricing problem (1) turns out to be a system of first-order ordinary differential equations in the time-to-maturity variable τ:

$$\begin{cases} -A' - A\kappa & = & \sigma_\xi \phi_\xi + A\sigma_\chi \phi_\chi, \\ -B' + \mu_\xi + \frac{1}{2}\sigma_\xi^2 + A\sigma_\xi\sigma_\chi\rho_{\xi\chi} + \frac{1}{2}A^2\sigma_\chi^2 & = & \sigma_\xi\lambda_\xi + A\sigma_\chi\lambda_\chi, \\ A(0) & = & 0, \\ B(0) & = & 0. \end{cases}$$

Its solution is given by (2) and (3). This completes the proof.

Importantly, the exposure of the market price of ξ-type risk to the transitory component χ_t ($\phi_\xi \neq 0$) implies that, even if deprived of full unit-root persistence ($\kappa > 0$ and $\kappa + \sigma_\chi\phi_\chi > 0$), χ_t has a futures-price impact that does not vanish as the delivery date diverges ($\tau \to +\infty$):

$$A(\infty) = -\frac{\sigma_\xi \phi_\xi}{\kappa + \sigma_\chi \phi_\chi}.$$

The next section visualizes and discusses the additional impact of state-dependent risk premia on the term structure of the futures prices.

3. Term-structure patterns

The analysis requires the expected spot price in τ years from now, which Schwartz and Smith (2000) work out to be (in log levels)

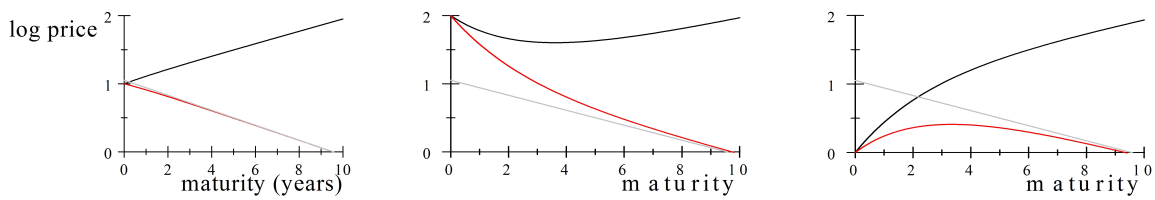
$$\ln E_t^{\mathbb{P}}[S_{t+\tau}]) \;=\; \xi_t \;+\; \chi_t e^{-\kappa\tau} \;+\; \left(\mu_\xi + \frac{\sigma_\xi^2}{2}\right)\tau \;+\; \frac{\sigma_\chi^2}{4\kappa}\left(1 - e^{-2\kappa\tau}\right) \;+\; \frac{\sigma_\xi\sigma_\chi\rho_{\xi\chi}}{\kappa}\left(1 - e^{-\kappa\tau}\right) \quad .$$

It will be plotted in black in the following figures. Another important pricing benchmark is the futures price prevaling at distant delivery dates, which is (in log levels)

$$\xi_t \;+\; \chi_t A(\infty) \;+\; D\tau \;+\; G \;+\; H \;.$$

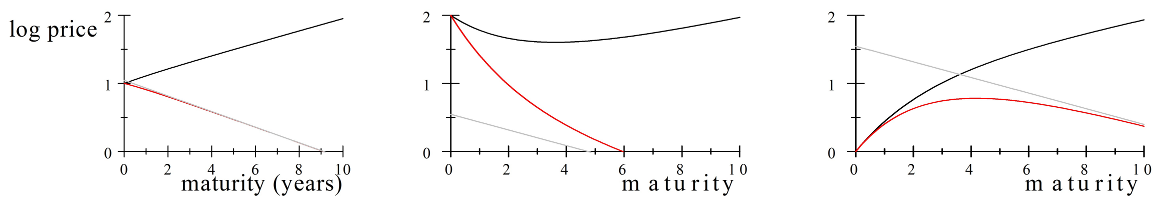
It will be plotted in grey. The futures log price $\ln F(\xi_t, \chi_t, \tau)$ will be plotted in red. I fix $\mu_\xi = 7\%$, $\sigma_\xi = 20\%$, $\kappa = 0.4$ (that is a "half-life" of the transitory component χ_t of about 21 months under $\mathbb{P}$), $\sigma_\chi = 15\%$, and $\rho_{\xi\chi} = 0.5$. The permanent component ξ_t of the spot log price is normalized to 1.

Figure 1



$\chi_t = 0$ $\chi_t = 1$ $\chi_t = -1$

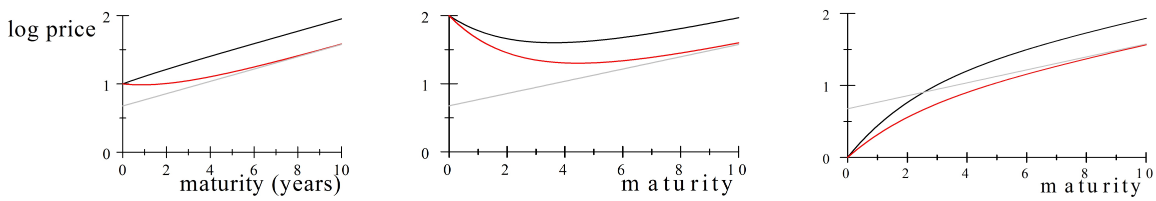
Figure 2



$\chi_t = 0$ $\chi_t = 1$ $\chi_t = -1$

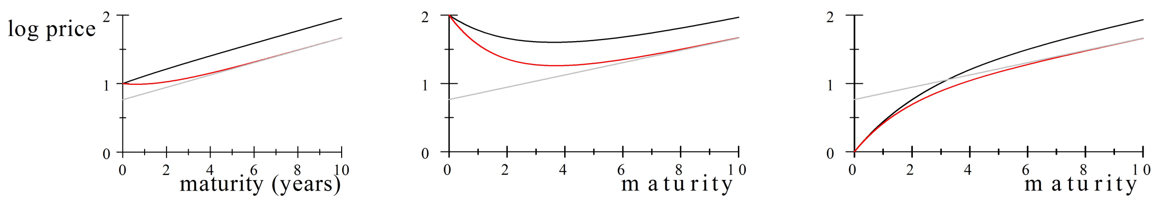
I begin with focusing on the pricing impact of $\Lambda_{\xi,t}$. Figure 1 shows the term-structure implications of the original Schwartz and Smith (2000) model with $\phi_\xi = \phi_\chi = \lambda_\chi = 0$ and $\lambda_\xi = 1$. The positive risk premium implies *normal backwardation* (i.e. $\ln E_t^{\mathbb{P}}[S_{t+\tau}]) > \ln F(\xi_t, \chi_t, \tau)$ for $\tau > 0$) and its size ($D < 0$) generally causes *backwardation* (i.e. $\ln F(\xi_t, \chi_t, \tau)$ decreases with τ) but for large negative transitory deviations from ξ_t, which prompt *contango* over the short-to-medium maturity dates (i.e. $\ln F(\xi_t, \chi_t, \tau)$ increases there with τ). Figure 2 visualizes the effect of switching on the state-dependent nature of $\Lambda_{\xi,t}$. Given $\phi_\xi = 1$, the changes in the slope D of the long-term futures log price and in its constant-intercept terms G and H are not substantial. What makes the difference is χ_t's long-run futures-price impact $A(\infty)$, which loads the state χ_t in the intercept of the long-term futures log price. Large positive transitory deviations foster a stronger backwardation ($\chi_t A(\infty) < 0$), whereas large negative deviations strengthen the contango over the short-to-medium maturity dates ($\chi_t A(\infty) > 0$).

Figure 3



$\chi_t = 0$ $\chi_t = 1$ $\chi_t = -1$

Figure 4



$\chi_t = 0$ $\chi_t = 1$ $\chi_t = -1$

I now turn to the pricing impact of $\Lambda_{\chi,t}$. Figure 3 depicts the term-structure implications of the original Schwartz and Smith (2000) model with $\phi_\xi = \phi_\chi = \lambda_\xi = 0$ and $\lambda_\chi = 1$. Again, the positive risk premium brings about normal backwardation. However, the slope D of the long-term futures log price is only slightly affected by λ_χ and remains positive, generating long-term contango. Backwardation over the short-to-medium maturity dates stems only from a large positive χ_t. Figure 4 shows that activating the state-dependent nature of $\Lambda_{\xi,t}$ ($\phi_\chi = 1$) has a subdued impact on the term structure of futures prices, the main change being their faster convergence toward the long-term benchmark ($\kappa + \sigma_\chi \phi_\chi > \kappa$).

4. Conclusions

For a generic commodity, I work out the proof of the closed-form extension of the celebrated Schwartz and Smith (2000) model of spot/futures pricing to state-dependent risk premia and point out that state dependence in the market price of the permanent spot-price risk plays an important role in shaping the term structure of futures prices.

Conflict of Interests

The author declares that there is no conflict of interests.

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