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A CONTINUOUS-IN-TIME FINANCIAL MODEL

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Abstract. In this paper, we construct a continuous-in-time model which is designed to be used for the finances of public institutions. This model is based on using measures over time interval to describe loan scheme, reimbursement scheme and interest payment scheme; and, on using mathematical operators to describe links existing between those quantities. The consistency of the model with respect to the real world is illustrated using examples and its mathematical consistency is checked. Then the model is used on simplified examples in order to show its capability to be used to forecast consequences of a decision or to set out a financial strategy.

Keywords: continuous-in-time model; financial model; financial strategy.

2010 AMS Subject Classification: 62P05.

1. Introduction

For the time being, a discrete model of financial multiyear planning has been being used by the company MGDIS (http://www.mgdis.fr/). This model allows to set out annual and multiyear budgets for any organization and, in particular, for local communities. It has also been being marketed by MGDIS in the context of the local community finances.

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This model uses tables and generates outcomes in the form of tables. Each value in the tables is a synthetic value of a given quantity over a given period of time.

The implementation of the model to provide answers to a question consists first in setting the whole time period of interest associated with the question. Then, it consists in defining the periods of time and the quantities under consideration. A table is filled with every known (or forecasted) quantity associated with every period of time. The model allows to compute the quantities that are consequences of the known ones on the first period of time. Then, knowing the values of all quantities under consideration on the first period, the model allows to compute – on the second period of time – the values of the quantities that are consequences of the quantities on the first period of time and of the known quantities on this second period of time. And finally, the process goes on to compute the values of all quantities on the third period, on the fourth one, etc.

This way to proceed gives good results, works well and software tools implementing it help organizations to foresee the consequences of their decisions and to elaborate their strategies.
Nevertheless it has drawbacks. The first one is related to the fact that the definition of the periods of time at stake is done at the beginning of the process. Consequently, in the case when an implementation of the model is done using given periods of time and when the same model is needed, but with other periods of time, all the process needs to be redone starting from the ground. As this is expensive, this is commonly not done. And in practice, only one model, with one set of periods of time, is implemented.
The second drawback is linked with the using of tables itself. This imposes to compute the quantities on the first period of time before computing them on the second one etc. Hence, this makes the model hard to be used within an automatic strategy elaboration tool that needs to manage constrains and goals several years ahead.

A way to overcome those drawbacks consists in designing models of a new kind which are continuous-in-time, which involve mathematical objects like measures and densities and which use mathematical operators like derivation, integration and convolution.
As those models are continuous-in-time (see Merton [9] and Sundaresan [10]), the question

of the periods of time is not an issue while implementing the model. They can of course be introduced once all the measures are computed. In particular, the outcomes of the model can be reported on any set of periods of time without reimplementing it.
On another hand, as it manages measures, that are defined on the whole time period of interest, using operators that link them, the issue of computing things on a given year before computing things on the next year etc. is avoided. Then the question of establishing strategies may be expressed as a problem of optimization under constraints involving a fitness (see Hanselmann, Schrempf & Hanebeck [7]), mathematical operators and (if they exist) their inverses.

Setting out first models of that kind is the aim of the present paper. This is done with the constraint that the discrete model of financial multiyear planning can be interpreted as particular case of the continuous-in-time models built in the present paper. This is important in order to reuse works that were previously led using the discrete model in the new framework.

We refer to Frénod & Safa [2, 4, 3], which improve one of the continuous-in-time financial models built in the present paper incorporating in it elements of control theory in order to determine the optimal loan scheme that achieves desired goals and that satisfies imposed constraints.

The paper is organized as follows. In section 2, we introduce the models. There are the "financial model with constant rate", the "model with variable rate" (which is a simple evolution of the former one). There is also the "financial model with constant rate set at instants of borrowing" which needs a non-direct improvement of the financial model with constant rate. The mathematical consistency of the models is analyzed and the way to interpret them is explained. In section 5, the models are used on simplified problems in order to show how they can be used.

2. Financial model with constant rate

2.1. **Mathematical material.** In this section, we build the financial model which we will work with hereafter. This model is based on the use of measures. We will take as definition of measures the Radon measure slant that consists in saying that a measure on a close, bounded and non-empty interval $[t_{\mathtt{I}}, \Theta_{\max}] \subset \mathbb{R}$ is a continuous and linear form acting on space

$\mathscr{C}_c^o([t_{\mathtt{I}},\Theta_{\max}])$ of continuous functions defined over $[t_{\mathtt{I}},\Theta_{\max}]$. It is well known that the usual norm on $\mathscr{C}_c^o([t_{\mathtt{I}},\Theta_{\max}])$ is:

$$\|\psi\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}])}=\sup_{t\in[t_{\mathtt{I}},\Theta_{\max}]}\big\{|\psi(t)|\big\}, \tag{2.1}$$

and that the set of Radon measures $\mathscr{M}([t_{\mathtt{I}},\Theta_{\max}])$ is a Banach space when provided with norm:

$$\|\mu\|_{\mathscr{M}((t_{\mathtt{I}},\Theta_{\max}))}=\sup_{\psi\in\mathscr{C}_c^o([t_{\mathtt{I}},\Theta_{\max}]),\psi\neq 0}\left\{\frac{\langle\mu,\psi\rangle}{\|\psi\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}])}}\right\}. \tag{2.2}$$

As we will use the convolution and Fourier Transform which are operators acting on measures over $\mathbb{R}$, we will also consider that $\mathscr{M}([t_{\mathtt{I}},\Theta_{\max}])$ is the set of Radon Measures over $\mathbb{R}$, supported in $[t_{\mathtt{I}},\Theta_{\max}]$.

Among $\mathscr{M}([t_{\mathtt{I}},\Theta_{\max}])$, some measures are absolutely continuous with respect to the Lebesgue measure dt. This means that they read $\rho(t)dt$, where t is the variable in $\mathbb{R}$. From the application point of view, the density $\rho(t)$ of such a measure can be interpreted as a time density (time density of borrowed amount linked with a loan, time density of repayment, *etc.*). In the sequel, we will call those kinds of measures "density measures". Other measures, for instance Dirac masses $\delta_{t=t_1}$ are concentrated. They can be interpreted as localized actions or payment *etc.*

Before explaining the quantities that are involved in the models and the relations between them, we introduce $T_{\min}$ which is the time scale below which nothing coming from the model will be observed. To be more precise, we say that a measure $\tilde{m}$ is observed over time interval $[t_1,t_2]$ if

$$\int_{t_1}^{t_2}\tilde{m}, \tag{2.3}$$

is computed. And, we will always, choose times t_1 and t_2 such that $t_2-t_1>T_{\min}$.

2.2. **Model building.** In what follows, $[t_{\mathtt{I}},\Theta_{\max}]$ stands for a time interval over which things happen.

The first quantity is the Loan Measure $\tilde{\kappa}_E$. It is defined such that the amount borrowed between times t_1 and t_2 is:

$$\int_{t_1}^{t_2} \tilde{\kappa}_E. \tag{2.4}$$

It is related to the Total Borrowed Amount of the loan: $\mathscr{K}_{max}$ by the following expression:

$$\int_{-\infty}^{+\infty} \tilde{\kappa}_E = \mathscr{K}_{max}. \tag{2.5}$$

The second one is the Capital Repayment Measure $\tilde{\rho}_{\mathscr{K}}$. It gives the scheme according to which the capital is repaid and it is defined such that the amount of capital which is repaid between t_1 and t_2 is:

$$\int_{t_1}^{t_2} \tilde{\rho}_{\mathscr{K}}. \tag{2.6}$$

Loan Measure $\tilde{\kappa}_E$ and Capital Repayment Measure $\tilde{\rho}_{\mathscr{K}}$ are both supposed to be in $\mathscr{M}([t_{\mathtt{I}}, \Theta_{\max}])$ and are connected by a convolution operator:

$$\tilde{\rho}_{\mathscr{K}} = \tilde{\kappa}_E \star \tilde{\gamma}, \tag{2.7}$$

where the Repayment Pattern $\tilde{\gamma}$ is a non-negative measure with total mass which equals 1, i.e:

$$\int_{-\infty}^{+\infty} \tilde{\gamma} = 1, \tag{2.8}$$

which support is included in $[0, \Theta_{\max} - t_{\mathtt{I}}]$ and which is such that the support of $\tilde{\kappa}_E \star \tilde{\gamma}$ is included in $[t_{\mathtt{I}}, \Theta_{\max}]$. The Repayment Pattern expresses the way an amount 1 borrowed at $t = 0$ is repaid. Notice that if $\tilde{\kappa}_E = \kappa_E(t)dt$ and $\tilde{\gamma} = \gamma(t)dt$ are density measures, then so is $\tilde{\rho}_{\mathscr{K}}$, i.e $\tilde{\rho}_{\mathscr{K}} = \rho_{\mathscr{K}}(t)dt$ and:

$$\rho_{\mathscr{K}}(t) = (\kappa_E \star \gamma)(t) = \int_{-\infty}^{+\infty} \kappa_E(s)\gamma(t-s)\, ds. \tag{2.9}$$

Loan Density κ_E and Repayment Density $\rho_{\mathscr{K}}$ are time densities. Since the amounts are expressed in monetary unit, the unit of those densities is monetary unit over time. Then, since the dimension of dt is time, $\kappa_E(t)dt$ and $\rho_{\mathscr{K}}(t)dt$ have the dimension of a monetary amount. By the way, because of its definition the dimension of density $\gamma(t)$ is the inverse of a time yielding $\gamma(t)dt$ has no dimension. This makes that formula (2.9) is homogeneous with regard to dimension issues. In the case when the measures are not density measures, then $\tilde{\kappa}_E$ and $\tilde{\rho}_{\mathscr{K}}$ have the dimension of monetary amount and $\tilde{\gamma}$ has no dimension. Thus, formula (2.7) is also consistent.

In order to illustrate formula (2.7) (or (2.9)), in Figure 1, we draw examples of $\tilde{\kappa}_E$, $\tilde{\gamma}$ and $\tilde{\rho}_{\mathscr{K}}$. The picture at the top left of Figure 1 shows three measures that are density measures. The function at the top is Density γ and the one in the middle is Density κ_E. The result of the convolution, which definition here is given by (2.9) is drawn at the bottom. The result of the action of γ on κ_E via the convolution is a shift toward right and an enlargement of the support. The picture at the top right shows the action of Repayment Pattern γ (which is a density measure) on a Loan Measure which is the Dirac mass $\delta_{t=\frac{1}{4}}$. We see that the result is a function which is a shift of $\frac{1}{4}$ of γ toward the right. This illustrates that the Repayment Pattern gives the way that an amount 1 borrowed at time 0 is repaid. Indeed, in the present situation the considered Loan Measure means that an amount 1 is borrowed at time $t = \frac{1}{4}$. Hence, Repayment Density $\rho_{\mathscr{K}}$ is the repayment density which is generated by an amount 1 borrowed at $t = 0$ shifted of $\frac{1}{4}$. The picture at the bottom left represents the convolution of Repayment Pattern which is the Dirac mass $\delta_{t=\frac{1}{4}}$ and density measure of Loan Density κ_E. The result is a function which is a shift of κ_E of $\frac{1}{4}$. Here, Repayment Measure $\tilde{\rho}_{\mathscr{K}}$ is a density measure. We notice the results of pictures in the top right and the bottom left are the same. This results from convolution property.
The picture at the bottom right shows the action of $\tilde{\gamma}$ which is Dirac mass $\delta_{t=\frac{1}{4}}$ on a sum of two Dirac $(\delta_{t=\frac{1}{4}} + \delta_{t=\frac{1}{2}})$. This convolution result is not density measure, it is $\tilde{\kappa}_E$ translated of $\frac{1}{4}$ which is the time at which the Repayment Pattern is concentrated.

The third quantity involved in the model is the Current Debt Field $\mathscr{K}_{RD}$. It is a function that, at any time t, gives the capital amount still to be repaid. The fourth one is the Interest Payment Measure $\tilde{\rho}_{\mathscr{I}}$. It is related to the Current Debt Field by a proportionality relation. This induces that $\tilde{\rho}_{\mathscr{I}} = \rho_{\mathscr{I}}(t)dt$ is always a density measure and that $\rho_{\mathscr{I}}(t)$ is linked with $\mathscr{K}_{RD}$ by:

$$\rho_{\mathscr{I}}(t) = \alpha \mathscr{K}_{RD}(t), \tag{2.10}$$

where, α is the loan rate, it is the inverse of a time (see Remark 2.1). Relation (2.10) is dimensionally homogeneous. Indeed, it relates Interest Payment Density $\rho_{\mathscr{I}}$, which dimension is the one of monetary amount over time, to field $\mathscr{K}_{RD}$ expressed in monetary unit by the multiplication by α which unit is the inverse of a time.

There are also other measures which are used in this model such as $\tilde{\rho}^{\mathtt{I}}_{\mathscr{K}}$, $\tilde{\sigma}$, $\tilde{\beta}$, $\tilde{\sigma}_g$. They are defined as follows. Measure $\tilde{\rho}^{\mathtt{I}}_{\mathscr{K}}$ is the Initial Debt Repayment Plan. It expresses how current debt amount at the initial instant will be repaid. It is a repayment scheme and it satisfies:

$$\int_{t_{\mathtt{I}}}^{+\infty} \tilde{\rho}^{\mathtt{I}}_{\mathscr{K}} = \mathscr{K}_{RD}(t_{\mathtt{I}}), \tag{2.11}$$

where $\mathscr{K}_{RD}(t_{\mathtt{I}})$ is the known Current Debt at initial time $t_{\mathtt{I}}$.
Algebraic Spending Measure $\tilde{\sigma}$ is defined such that the difference between spendings and incomes required to satisfy the current needs between times t_1 and t_2 is:

$$\int_{t_1}^{t_2} \tilde{\sigma}. \tag{2.12}$$

Measure of Isolated Spending $\tilde{\beta}$ is related to the notion of project. We consider that, in the budget, some spendings can be gathered because they all contribute to a common goal. We call this gathering a project and $\tilde{\beta}$ is the Spending Scheme associated to the project. The last considered measure is the Current Spending $\tilde{\sigma}_g$. It is the Scheme associated to spendings which are not related to the project and it is defined as:

$$\tilde{\sigma}_g = \tilde{\sigma} - \tilde{\beta}. \tag{2.13}$$

In order to balance the budget in terms of incomes and spendings, the budget balance rules obligate Loan Measure $\tilde{\kappa}_E$ to be equal to Measure of Algebraic Spending $\tilde{\sigma}$ added to measures associated with quantities that have to be repaid or paid. This is expressed by the following equality:

$$\tilde{\kappa}_E = \tilde{\sigma} + \tilde{\rho}_{\mathscr{K}} + \tilde{\rho}^{\mathtt{I}}_{\mathscr{K}} + \tilde{\rho}_{\mathscr{I}}. \tag{2.14}$$

According to (2.13) and (2.14), the dimension of the following measures $\tilde{\rho}^{\mathtt{I}}_{\mathscr{K}}$, $\tilde{\sigma}$, $\tilde{\beta}$, $\tilde{\sigma}_g$ is the one of a monetary amount.

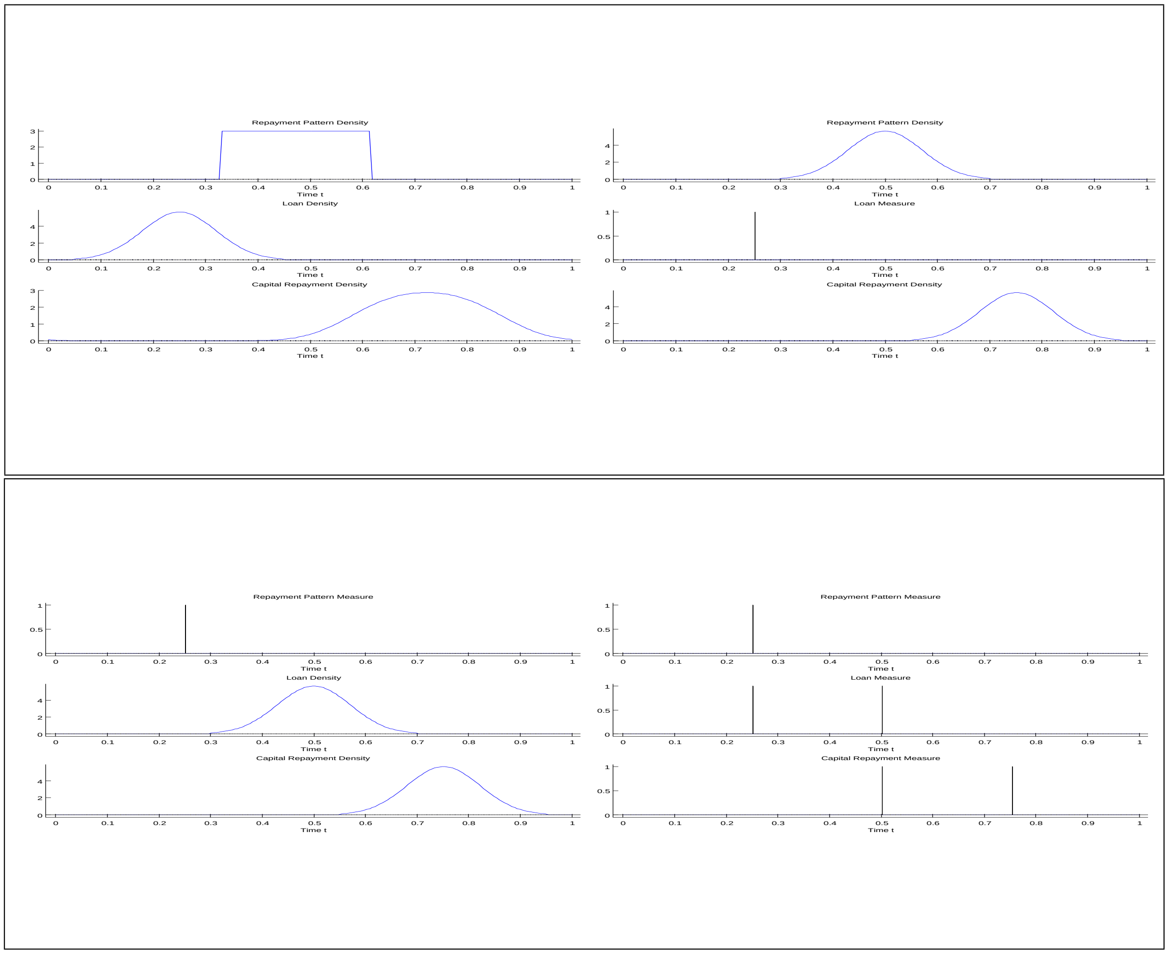

FIGURE 1. In each of the four pictures, $\tilde{\gamma}$ is represented at the top, $\tilde{\kappa}_E$ in the middle and $\tilde{\rho}_{\mathscr{K}} = \tilde{\kappa}_E \star \tilde{\gamma}$ at the bottom. (When the considered measure is a density measure, its density is drawn. When it undergoes concentrations, bars located at the concentration positions, with hight the amount which is concentrated at each location, are drawn).

Remark 2.1. *Rate α can be written in terms of the usual rate of the loan τ. In fact, τ is determined such that, if the Current Debt $\mathscr{K}_{RD}$ remains the same and equals a constant $\overline{K}$ over the period $[t_1, t_1 + 1year]$,*

$$\int_{t_1}^{t_1+1year} \tilde{\rho}_{\mathscr{I}} = \tau \overline{K}, \tag{2.15}$$

which means that the interest paid over the year is equal to $\tau\overline{K}$*. In this case,* $\rho_{\mathscr{I}}(t) = \alpha\overline{K}$ *and then relation* $\rho_{\mathscr{I}}(t) = \alpha\mathscr{K}_{RD}(t)$ *becomes:*

$$(1year)\alpha\overline{K} = \tau\overline{K}. \tag{2.16}$$

So:

$$\alpha = \frac{\tau}{(1year)}. \tag{2.17}$$

It is important to notice that τ *has no dimension (as it links variables that express both in monetary units). The variable* α *is a quantity that is dimensionless divided by a time. It has the dimension of an inverse of time.*

The Current Debt Field $\mathscr{K}_{RD}$ is related to Loan Measure $\tilde{\kappa}_E$ and Repayment Measure $\tilde{\rho}_{\mathscr{K}}$ by the following Ordinary Differential Equation:

$$\frac{d\mathscr{K}_{RD}}{dt} = \kappa_E(t) - \rho_{\mathscr{K}}(t) - \rho^{\mathtt{I}}_{\mathscr{K}}(t). \tag{2.18}$$

The solution of this ODE is expressed:

$$\mathscr{K}_{RD}(t) = \mathscr{K}_{RD}(t_{\mathtt{I}}) + \int_{t_{\mathtt{I}}}^{t} \tilde{\kappa}_E - \int_{t_{\mathtt{I}}}^{t} \tilde{\rho}_{\mathscr{K}} - \int_{t_{\mathtt{I}}}^{t} \tilde{\rho}^{\mathtt{I}}_{\mathscr{K}} = \int_{t_{\mathtt{I}}}^{t} \tilde{\kappa}_E - \int_{t_{\mathtt{I}}}^{t} \tilde{\rho}_{\mathscr{K}} + \int_{t}^{+\infty} \tilde{\rho}^{\mathtt{I}}_{\mathscr{K}}. \tag{2.19}$$

Using expression (2.7) of Repayment Measure $\tilde{\rho}_{\mathscr{K}}$ and expression (2.10), the Interest Payment Density $\rho_{\mathscr{I}}$ can be expressed in terms of Loan Measure $\tilde{\kappa}_E$:

$$\rho_{\mathscr{I}}(t) = \alpha \int_{t_{\mathtt{I}}}^{t} \tilde{\kappa}_E - \alpha \int_{t_{\mathtt{I}}}^{t} \tilde{\kappa}_E \star \tilde{\gamma} + \alpha \int_{t}^{+\infty} \tilde{\rho}^{\mathtt{I}}_{\mathscr{K}}. \tag{2.20}$$

2.3. **Justifications and properties.** We will justify, in this section, the convolution relation (see (2.7) or (2.9)), that links Repayment Measure $\tilde{\rho}_{\mathscr{K}}$ with Loan Measure $\tilde{\kappa}_E$. We will adopt both mathematical and financial point of view.

The picture at the top left of Figure 2 shows the action of Repayment Pattern $\tilde{\gamma}$ which is a combination of seven Dirac masses (three Dirac masses have the same mass $\frac{5}{19}$ at different times

0.3, 0.4, 0.5, the other four Dirac masses have the same mass $\frac{1}{19}$ at times 0.25, 0.35, 0.45, 0.55, see top diagram) on Dirac mass located in 0.1 with mass equal to 20 (see middle diagram). This convolution result is not a density measure, it is $20 \times \tilde{\gamma}$ translated of 0.1 (see bottom diagram). This simulation can be interpreted from the financial slant as follows. The middle diagram means the Loan amount is borrowed at once at time 0.1. The measure drawn in the bottom diagram is computed using formula (2.7), it means that the total repayment is made of amount $\frac{100}{19}$ at times 0.4, 0.5, 0.6 and amount $\frac{20}{19}$ at times 0.35, 0.45, 0.55, 0.65. Thus, the total repayment amount is equal to $\frac{300+80}{19} = \frac{380}{19} = 20$, meaning that it equals the amount borrowed at time 0.1. This diagram represents well the Repayment Plan associated with localized loan which is represented by the middle diagram according to the Repayment Pattern $\tilde{\gamma}$ shown in the top diagram. This example illustrates that formula (2.7) models correctly the way to compute the Repayment Scheme from the loan.

The picture at the top right of Figure 2 represents the convolution of Repayment Pattern $\tilde{\gamma}$ which is the same as previously and Loan Measure $\tilde{\kappa}_E = 20\delta_{t=0.1} + 10\delta_{t=0.17}$ (see middle diagram). The result (see bottom diagram) is the superposition of two measures, the first measure is the one of the bottom diagram at the top left picture, and the second measure is this same one shifted of 0.07 and divided in half.
The meaning of this simulation can be given as follows. The middle diagram means that the loan is shared into two pieces, the first one consists in borrowing 20 at time 0.1 and the second one consists in borrowing 10 at time 0.17. The bottom diagram represents the result of formula (2.7). The total repayment is constitued by two repayments. The first repayment is associated with the first piece and is done according to the Repayment Pattern $\tilde{\gamma}$. The second repayment is associated with the second piece. The total repayment of the first piece is shown at the top left picture in the bottom diagram. The repayment associated with the second piece is made of amount $\frac{50}{19}$ at times 0.47, 0.57, 0.67, and of amount $\frac{10}{19}$ at times 0.42, 0.52, 0.62, 0.72. The total repayment is well the combination of these two repayments, which is equal to the following amount: $20 + (3 \times \frac{50}{19}) + (4 \times \frac{10}{19}) = 30$. We can also conclude with this example that formula

(7) is pertinent.

The picture at the bottom left of Figure 2 shows the action of Repayment Pattern $\tilde{\gamma}$, which is the same as the previous one, on Loan Measure $\tilde{\kappa}_E$ which is a combination of five Dirac masses (two Dirac masses have the same mass 20 at times 0.1 and 0.17, three Dirac masses have the same mass 10 at times 0.07, 0.13 and 0.2, see middle diagram). This convolution result is the combination of two measures. The first measure is the sum of $20\times\tilde{\gamma}$ which is translated of 0.1 and $20\times\tilde{\gamma}$ which is translated of 0.17. The second measure is the sum of the same quantity $10\times\tilde{\gamma}$ which are translated respectively of 0.07, 0.13 and 0.2.
The meaning of this simulation is the following. The middle diagram means that the loan is shared into five pieces, two pieces consist in borrowing 20 at times 0.1 and 0.17 and the other three pieces consist in borrowing 10 at times 0.07, 0.13 and 0.2. The total repayment is the sum of five repayments. Each repayment is associated with one piece and is done according to the Repayment Pattern $\tilde{\gamma}$. The total repayment can be computed as previously as the combination of these five repayments. The bottom diagram represents the result of formula (2.7) which models correctly the way to compute the Repayment Plan from the loans.

The bottom right picture in this same Figure 2 shows three density measures. The function in the top diagram is Density γ and the one in the middle diagram is Density κ_E. They are such that γdt and $\kappa_E dt$ are respectively close to Repayment Pattern Measure $\tilde{\gamma}$ and to Loan Measure $\tilde{\kappa}_E$ of the bottom left picture. The result of the convolution is drawn in the bottom diagram. It is Repayment Density $\rho_{\mathscr{K}}$ and it is such that $\rho_{\mathscr{K}}dt$ is close to $\tilde{\rho}_{\mathscr{K}}$ of the bottom diagram in the bottom left picture.

This illustrates that, if a Repayment Pattern Density γ and a Loan Density κ_E are respectively idealizations of Repayment Pattern Measure $\tilde{\gamma}$ and Loan Measure $\tilde{\kappa}_E$, then Repayment Density $\rho_{\mathscr{K}}$ given by (2.9) is an idealization of Repayment Measure $\tilde{\rho}_{\mathscr{K}}$ given by (2.7).

To express this in more mathematical terms, we notice that for any measures $\tilde{\kappa}_{E1}$, $\tilde{\kappa}_{E2}$ in $\mathscr{M}([t_{\mathtt{I}},\,\Theta_{\max}])$ and $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ in $\mathscr{M}([0,\Theta_{\max}])$ such that $\tilde{\kappa}_{E2}\star\tilde{\gamma}_2$ and $\tilde{\kappa}_{E1}\star\tilde{\gamma}_1$ are in $\mathscr{M}([t_{\mathtt{I}},\Theta_{\max}])$, we have:

$$\|\tilde{\kappa}_{E2}\star\tilde{\gamma}_2-\tilde{\kappa}_{E1}\star\tilde{\gamma}_1\|_{\mathscr{M}([t_{\mathtt{I}},\Theta_{\max}])}\leq\|\tilde{\kappa}_{E2}-\tilde{\kappa}_{E1}\|_{\mathscr{M}([t_{\mathtt{I}},\Theta_{\max}])}\|\tilde{\gamma}_1\|_{\mathscr{M}([0,\Theta_{\max}])}+ \|\tilde{\gamma}_2-\tilde{\gamma}_1\|_{\mathscr{M}([0,\Theta_{\max}])}\|\tilde{\kappa}_{E2}\|_{\mathscr{M}([t_{\mathtt{I}},\Theta_{\max}])}. \tag{2.21}$$

Using this formula with $\tilde{\kappa}_{E1}$ and $\tilde{\gamma}_1$, which are respectively the measures in the middle and the top diagram of the bottom left picture and with $\tilde{\kappa}_{E2}=\kappa_{E2}(t)dt$, $\tilde{\gamma}_2=\gamma_2(t)dt$, where κ_{E2}, γ_2 are the densities drawn in the middle and the top diagram of the bottom right picture, we obtain the following conclusion: measure $\tilde{\rho}_{\mathscr{K}1}$ which is the Repayment Plan associated with Loan Measure $\tilde{\kappa}_{E1}$ and with Repayment Pattern $\tilde{\gamma}_1$, and, measure $\rho_{\mathscr{K}2}$ which is the Repayment Density associated with Loan Density κ_{E2} and with Repayment Pattern Density γ_2 satisfy:

$$\|\rho_{\mathscr{K}2}(t)dt-\tilde{\rho}_{\mathscr{K}1}\|_{\mathscr{M}([t_{\mathtt{I}},\Theta_{\max}])}\leq\|\kappa_{E2}(t)dt-\tilde{\kappa}_{E1}\|_{\mathscr{M}}\|\tilde{\gamma}_1\|_{\mathscr{M}([0,\Theta_{\max}])}+ \|\gamma_2(t)dt-\tilde{\gamma}_1\|_{\mathscr{M}([0,\Theta_{\max}])}\|\kappa_{E2}(t)dt\|_{\mathscr{M}([t_{\mathtt{I}},\Theta_{\max}])}. \tag{2.22}$$

This means that if $\tilde{\kappa}_{E1}$ and $\tilde{\kappa}_{E2}$ are close and if $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are close, then, $\tilde{\rho}_{\mathscr{K}2}$ and $\tilde{\rho}_{\mathscr{K}1}$ are also close.

We just saw that if any two density measures are respectively close to two concentrated measures, then, the density measure which is obtained by the convolution of these two density measures is also close to the concentrated measure which is obtained by the convolution of these two concentrated measures.

This will be the basis that will allow us to prefer using density measures in the models and simulations. In fact, there are several mathematical and financial reasons for this preference. From the mathematical point of view, it is clearly easier to handle functions than measures. Indeed, with functions, we can for instance use the Hilbert nature of L^2 spaces, we can call upon functional analysis and numerical analysis. From the financial point of view, the density approach can bring fuzziness, for instance in front of uncertainty, that can be welcome.

Loan and Repayment Measures are represented by concentrated measures in the left pictures and the top right picture of Figure 2. This formalism is consistent with reality. Indeed, in the real world, amounts are borrowed at fixed times and amounts are also repaid at fixed times. For

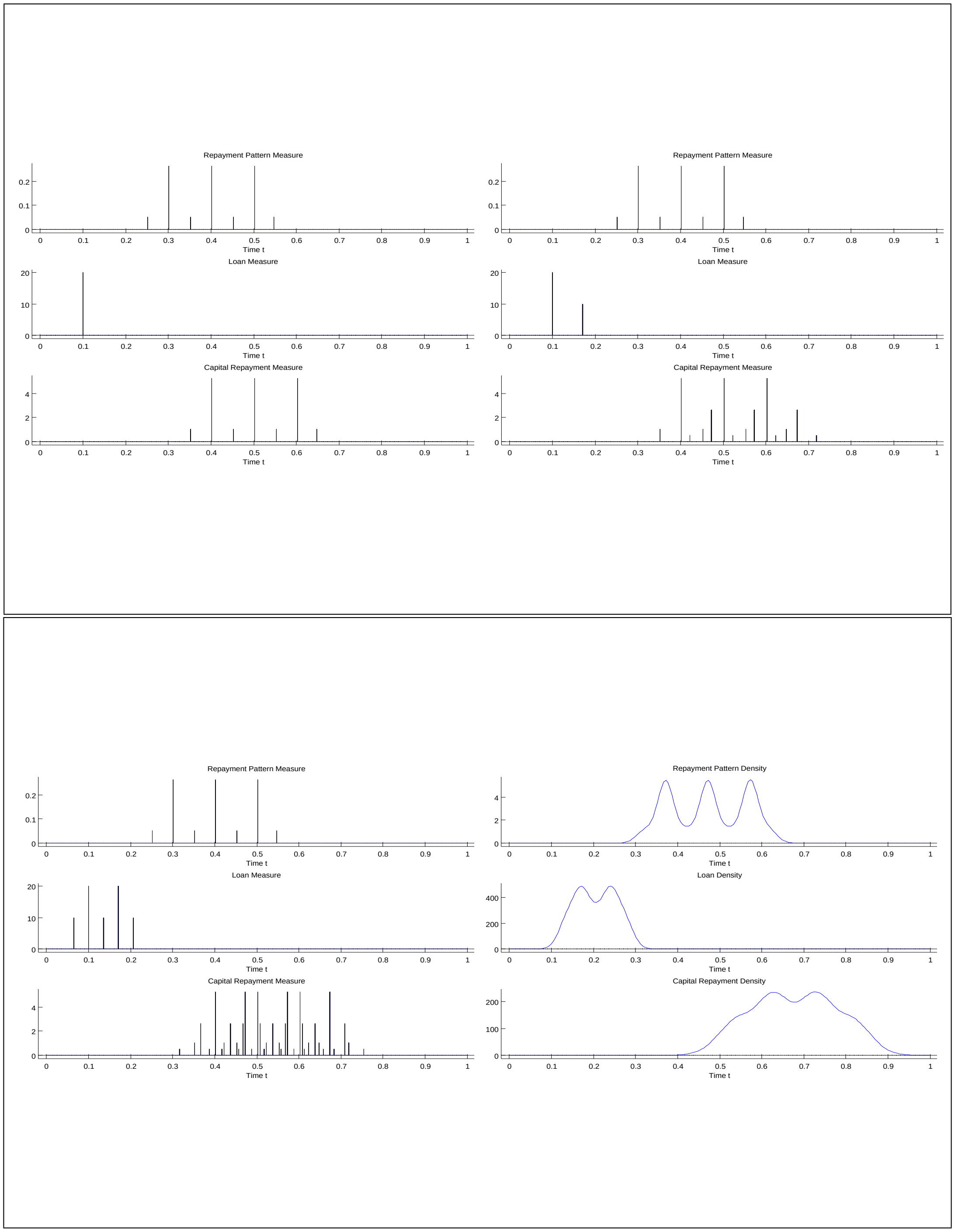

FIGURE 2. In each of the four pictures, $\tilde{\gamma}$ is represented at the top, $\tilde{\kappa}_E$ in the middle and $\tilde{\rho}_{\mathscr{K}} = \tilde{\kappa}_E \star \tilde{\gamma}$ at the bottom. These measures are concentrated measures in pictures at the left and in the top right picture. They are density measures in the bottom right picture.

instance, we can easily interpret the top left picture of Figure 2 by saying that an amount is borrowed at time 0.1 and repaid in seven times (at times 0.35, 0.4, 0.45, 0.5, 0.55, 0.6, 0.65). It is concluded that financial simulation using concentrated measures gives realistic results.
Loan and Repayment Measures that are represented in the bottom right picture of Figure 2 are density measures. A direct interpretation of the diagrams of this picture leads that amounts are borrowed continuously in time and amounts are also repaid continuously in time. This is not in accordance with the real world.

In order to describe how the simulation in the bottom right picture of Figure 2 is linked with the simulation of the bottom left picture of this same figure, a Friedrich mollifier is introduced. A Friedrich mollifier is a regular non-negative even function such that its integral is equal to 1. We take as Friedrich mollifier a Gaussian function.
The picture at the bottom right represents three diagrams. The one at the top is the density γ of measure γdt which is the convolution of this Friedrich mollifier with Repayment Pattern Measure $\tilde{\gamma}$ which is directly on its left (in the bottom left picture). The one in the middle is the density κ_E of measure $\kappa_E dt$ which is the convolution of this Friedrich mollifier with Loan Measure $\tilde{\kappa}_E$ which is directly on its left. The diagram at the bottom of this figure represents Repayment Density $\rho_{\mathscr{K}}$ which is related to γ and κ_E by equality (2.9).
Observing the bottom right picture and the bottom left picture of Figure 2 brings the conclusion that the integral over a large enough interval of Loan Measure $\tilde{\kappa}_E$ and Repayment Measure $\tilde{\rho}_{\mathscr{K}}$ give results which are respectively close to the integral over the same interval of density measures $\kappa_E dt$ and $\rho_{\mathscr{K}} dt$. In other words, if the simulations of the bottom pictures are observed (see (2.3)) over a large enough interval, they bring very close conclusions.

This interpretation, calling upon ”model observation”, may justify that we can use density measures in place of concentrated measures in the models and simulations.

Moreover, the bottom right picture in Figure 2 can be seen as a probabilistic version of the bottom left picture of Figure 2. This probabilistic approach is related to an approximation, which allows to replace pertinently a concentrated measure with concentrations that are not deterministically known by a density measure. Indeed, if for instance, we imagine that the instants where the measure of the bottom diagram at the bottom left picture is concentrated are not known precisely, then, this measure can be replaced by the one of the bottom diagram of the bottom right picture.

In order to deepen this probabilistic vision, several references are proposed to the reader. The paper of Hanselmann, Schrempf & Hanebeck [7] describes a numerical method which is based on optimization and estimate techniques including this approach. The target of this numerical method is to produce a good estimate of a density measure which is related only to the information about a concentrated measure. This built density measure is considered as the right estimator and the optimal approximation of the considered concentrated measure. Section 3 of [7] is referred for more technical details in the optimization part.
The paper of Klumpp & Hanebeck [8] proposes an approach which allows the multiplication of two Dirac measures which is mathematically not defined. The result of this multiplication is approached by density measure. The numerical iterative method giving this approximation is given in detail in section 5 of this paper. In this same section, Figures 6, 7 and 8 show the result of this approximation.

Beside this probabilistic vision, we can introduce a second one which is related to the mean field limit theory (see Benaim & Le Boudec [1] and Gast, Gaujal & Le Boudec [5]), for instance used in the kinetic theory of gases (see Golse [6]). The framework of this second vision is when borrower borrows and repays very often small amounts. It is based on the mathematical approach which describes the behavior of a concentrated measure when the number of instants at which the measure is concentrated goes to infinity while the mass at each instant of concentration goes to 0. The framework we consider, brings us to model loan and repayment by sums of Dirac masses like $\sum_{i=1}^{N} \omega_i^N \delta_{t=t_i^N}$, where $N \to +\infty$ while $\sum_{i=1}^{N} \omega_i^N$ is bounded independently of N. In order to avoid concentration of the Dirac mass locations, we also assume $\forall [a,b] \subset \mathbb{R}$, $\mathrm{Card}\{i, t_i^N \in [a,b]\}$ to be also bounded by $C(b-a)N$, where C is a constant independent of N. Those sums of Dirac masses are then approximated by density measures. In this framework, stability formula (2.22) insures that, if this approximation is done for Loan Measure $\tilde{\kappa}_{E1}$ (approximated by $\kappa_{E2}dt$) and for Repayment Pattern $\tilde{\gamma}_1$ (approximated by $\gamma_2 dt$), Repayment Density $\rho_{\mathscr{K}2} = \kappa_{E2} \star \gamma_2$ is such that $\rho_{\mathscr{K}2}$ approximates well Repayment Measure $\tilde{\rho}_{\mathscr{K}1} = \tilde{\kappa}_{E1} \star \tilde{\gamma}_1$.

3. Mathematical properties of the model

In this section we explore mathematical properties of the model we built in the previous section. Those properties will be useful for some aspects of the model implementation to come in the following. For our purpose and to be able to use some specific mathematical tools, we consider the case when all measures are density measures.

For any interval $[t_1,t_2]$, $t_2 > t_1$, $\mathbb{L}^2([t_1,t_2])$ stands for the space of square-integrable functions over $\mathbb{R}$ having their support in $[t_1,t_2]$. For a positive number Θ_γ such that $\Theta_\gamma < \Theta_{\max} - t_{\mathtt{I}}$, we set the Repayment Pattern γ such that:

$$\gamma \in \mathbb{L}^2([0,\Theta_\gamma]). \tag{3.1}$$

In order to achieve what is described after equation (2.8), we need to choose Loan Density κ_E with support being included in $[t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma]$.

3.1. General results. Using relations (2.9), (2.14) and (2.20), we can directly state the following theorem.

Theorem 3.1. *If all measures of model* ((2.4)-(2.14), (2.18)-(2.20)) *are density measures, if Repayment Pattern γ satisfies relation* (3.1) *and if Loan Density κ_E is in* $\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max} - \Theta_\gamma])$ *and Initial Debt Repayment Density $\rho^{\mathtt{I}}_{\mathscr{K}}$ is in* $\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])$ *then Algebraic Spending Density σ is also in* $\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])$ *and has the next expression in terms of Loan Density κ_E:*

$$\sigma(t) = \kappa_E(t) - (\kappa_E \star \gamma)(t) - \alpha \int_{t_{\mathtt{I}}}^{t} (\kappa_E - \kappa_E \star \gamma)(s)ds \; - \alpha \int_{t}^{\Theta_{\max}} \rho^{\mathtt{I}}_{\mathscr{K}}(s)ds \; - \rho^{\mathtt{I}}_{\mathscr{K}}(t). \tag{3.2}$$

Relation (3.2) is generally not invertible. Yet, it is possible to obtain an expression that allows, when it makes sense, to compute Loan Density κ_E from Algebraic Spending Density σ. For doing this, in a first place, defining linear operator $\mathscr{L} : \mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max} - \Theta_\gamma]) \to \mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])$ (acting on Loan Density κ_E) by

$$\mathscr{L}[\kappa_E](t) = (\kappa_E - \kappa_E \star \gamma)(t) - \alpha \int_{t_{\mathtt{I}}}^{t} (\kappa_E - \kappa_E \star \gamma)(s)\, ds, \tag{3.3}$$

and operator $\mathscr{D} : \mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}]) \to \mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ (acting on Initial Debt Repayment Density $\rho^{\mathtt{I}}_{\mathscr{K}}$) by

$$\mathscr{D}[\rho^{\mathtt{I}}_{\mathscr{K}}](t) = -\alpha \int_t^{\Theta_{\max}} \rho^{\mathtt{I}}_{\mathscr{K}}(s)\, ds - \rho^{\mathtt{I}}_{\mathscr{K}}(t), \tag{3.4}$$

expression (3.2) of density σ reads

$$\sigma(t) = \mathscr{L}[\kappa_E](t) + \mathscr{D}[\rho^{\mathtt{I}}_{\mathscr{K}}](t). \tag{3.5}$$

Secondly, we state the following lemma linking the Fourier Transforms of κ_E, γ and $\mathscr{L}[\kappa_E]$.

Lemma 3.2. *If function κ_E is in $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$ and if γ satisfies relation* (3.1)*, then we have the following equality:*

$$(1 - \mathscr{F}(\gamma))\mathscr{F}(\kappa_E) = \mathscr{F}\Big(\mathscr{L}[\kappa_E] + \alpha \int_{t_{\mathtt{I}}}^{\bullet} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(\bullet - s)}\, ds\Big), \tag{3.6}$$

where $\mathscr{F}$ stands for the Fourier Transform Operator, where operator $\mathscr{L}$ is defined by (3.3) *and where*

$$\mathscr{F}\Big(\int_{t_{\mathtt{I}}}^{\bullet} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(\bullet - s)}\, ds\Big) \tag{3.7}$$

stands for the Fourier Transform of function

$$t \mapsto \int_{t_{\mathtt{I}}}^{t} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(t - s)}\, ds. \tag{3.8}$$

Proof. Integrating by parts states that:

$$\begin{aligned} &\int_{t_{\mathtt{I}}}^{t} \left(\int_{t_{\mathtt{I}}}^{s} (\kappa_E - \kappa_E \star \gamma)(y)\, dy \right) \times \alpha \mathrm{e}^{\alpha(t-s)}\, ds \\ &\qquad = \int_{t_{\mathtt{I}}}^{t} (\kappa_E - \kappa_E \star \gamma)(s) \times \mathrm{e}^{\alpha(t-s)}\, ds - \int_{t_{\mathtt{I}}}^{t} (\kappa_E - \kappa_E \star \gamma)(s)\, ds. \end{aligned} \tag{3.9}$$

From this, we get the following equality:

$$\int_{t_{\mathtt{I}}}^{t} (\kappa_E - \kappa_E \star \gamma)(s)\, ds = \int_{t_{\mathtt{I}}}^{t} \left((\kappa_E - \kappa_E \star \gamma)(s) - \alpha \int_{t_{\mathtt{I}}}^{s} (\kappa_E - \kappa_E \star \gamma)(y)\, dy \right) \times \mathrm{e}^{\alpha(t-s)}\, ds. \tag{3.10}$$

Using definition (3.3) of operator $\mathscr{L}$, equality in (3.10) is multiplied by α to give:

$$\alpha \int_{t_{\mathtt{I}}}^{t} (\kappa_E - \kappa_E \star \gamma)(s)\, ds = \alpha \int_{t_{\mathtt{I}}}^{t} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(t-s)}\, ds. \tag{3.11}$$

Replacing $\alpha\int_{t_{\mathtt{I}}}^{t}(\kappa_E-\kappa_E\star\gamma)(s)\,ds$ in relation (3.11) by $(\kappa_E-\kappa_E\star\gamma)(t)-\mathscr{L}[\kappa_E](t)$ which is possible because of (3.3), we obtain the following equality:

$$\kappa_E-\kappa_E\star\gamma=\mathscr{L}[\kappa_E]+\alpha\int_{t_{\mathtt{I}}}^{\bullet}\mathscr{L}[\kappa_E](s)\mathrm{e}^{\alpha(\bullet-s)}\,ds. \tag{3.12}$$

Applying Fourier Transform to relation (3.12), we obtain equality (3.6), proving the lemma. □

From Lemma 3.2 and equality (3.5) the following theorem can be stated.

Theorem 3.3. *If Repayment Pattern is a density measure with its density γ satisfying relation* (3.1)*, and if Initial Debt Repayment Density $\rho^{\mathtt{I}}_{\mathscr{K}}$ is in $\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])$; if we choose an Algebraic Spending Density σ in $\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])$ such that*

$$\frac{\mathscr{F}\Big(\sigma-\mathscr{D}[\rho^{\mathtt{I}}_{\mathscr{K}}]+\alpha\displaystyle\int_{t_{\mathtt{I}}}^{\bullet}\sigma(s)-\mathscr{D}[\rho^{\mathtt{I}}_{\mathscr{K}}](s)\,\mathrm{e}^{\alpha(\bullet-s)}\,ds\Big)}{1-\mathscr{F}(\gamma)}\in\mathbb{L}^2(\mathbb{R}), \tag{3.13}$$

then function κ_E given in terms of σ by:

$$\kappa_E=\mathscr{F}^{-1}\left(\frac{\mathscr{F}\Big(\sigma-\mathscr{D}[\rho^{\mathtt{I}}_{\mathscr{K}}]+\alpha\displaystyle\int_{t_{\mathtt{I}}}^{\bullet}\sigma(s)-\mathscr{D}[\rho^{\mathtt{I}}_{\mathscr{K}}](s)\,\mathrm{e}^{\alpha(\bullet-s)}\,ds\Big)}{1-\mathscr{F}(\gamma)}\right), \tag{3.14}$$

where $\mathscr{F}^{-1}$ stands for the inverse of the Fourier Transform, is in $\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])$ and is the Loan Density that brings Algebraic Spending Density σ using model ((2.4)-(2.14)*,* (2.18)-(2.20)*).*

This theorem is a generic result that gives indication on the way to reverse the process that consists in in computing Algebraic Spending Density σ from Loan Density κ_E into a process that consists in building a Loan Density κ_E in order to achieve a desired Algebraic Spending Density σ. Yet, condition (3.13) needs to be explained, as it is not very usable in this form. Nevertheless, It seems to be too complicated to give it a more usable form remaining at this level of genericallity. Hence, to give a more precise form to (3.13) we will choose a given expression of Repayment Pattern Density γ.

3.2. **Results with a specific Repayment Pattern.**

Theorem 3.4. *If*

$$\gamma = \frac{1}{\Theta_\gamma} \mathbb{1}_{[0,\Theta_\gamma]}, \tag{3.15}$$

and if Initial Debt Repayment Density $\rho^{\mathtt{I}}_{\mathscr{K}}$ *and Algebraic Spending Density* σ *are in* $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ *and satisfy the following equality:*

$$\int_{t_{\mathtt{I}}}^{\Theta_{\max}} \left(\sigma(y) - \mathscr{D}[\rho^{\mathtt{I}}_{\mathscr{K}}](y) + \alpha \int_{t_{\mathtt{I}}}^{y} \sigma(s) - \mathscr{D}[\rho^{\mathtt{I}}_{\mathscr{K}}](s) \mathrm{e}^{\alpha(y-s)}\, ds \right) dy = 0, \tag{3.16}$$

then, there exists an unique Loan Density κ_E *in* $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$ *(which expression is given by formula* (3.14)*) such that* (3.5) *holds.*

In order to prove this theorem, we need the following lemmas:

Lemma 3.5. *Under the same assumptions as in Theorem 3.4, linear operator* $\mathscr{L}$ *given by relation* (3.3) *is a one-to-one map.*

Proof. In a first place, according to equality (3.12), if Loan Density κ_E is in $\mathrm{Ker}(\mathscr{L})$, then it satisfies:

$$\kappa_E - \kappa_E \star \gamma = 0. \tag{3.17}$$

And, according to definition (3.3) of operator $\mathscr{L}$, if Loan Density κ_E satisfies relation (3.17), then, κ_E is in $\mathrm{Ker}(\mathscr{L})$. Consequently, we have:

$$\mathrm{Ker}(\mathscr{L}) = \left\{ \kappa_E \in \mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma]), \kappa_E - (\kappa_E \star \gamma) = 0 \right\}. \tag{3.18}$$

Secondly, for any positive integer n, we consider P_n the vector space of polynomial functions of degree n and restricted to interval $(t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma)$. We will show that

$$\mathrm{Ker}(\mathscr{L}) \cap \mathrm{P}_n = \{0\}. \tag{3.19}$$

In order to show relation (3.19), we will show that the coefficients of polynomial function κ_E are zero for each κ_E in $\mathrm{Ker}(\mathscr{L}) \cap \mathrm{P}_n$. If $\kappa_E \in \mathrm{Ker}(\mathscr{L}) \cap \mathrm{P}_n$, then, for any $x \in (t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma)$,

$$\kappa_E(x) = \left(\sum_{i=0}^{i=n} c_i x^i \right) \times \mathbb{1}_{[t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma]}(x), \tag{3.20}$$

where $(c_i)_{0\leq i\leq n}$ are its coefficients and

$$\kappa_E - \kappa_E \star \gamma = 0. \tag{3.21}$$

Using equalities (3.15) and (3.20), we obtain

$$(\kappa_E \star \gamma)(x) = \frac{1}{\Theta_\gamma}\sum_{i=0}^{i=n} c_i \int_{[t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma]\cap[x-\Theta_\gamma,x]} y^i\, dy. \tag{3.22}$$

Now, on the one hand, we show that coefficients $(c_i)_{0\leq i\leq n}$ are zero, if $t_{\mathtt{I}}+\Theta_\gamma \leq \Theta_{\max}-\Theta_\gamma$. In this case, for any x such that $t_{\mathtt{I}}+\Theta_\gamma \leq \Theta_{\max}-\Theta_\gamma < x < \Theta_{\max}$, x is outside the support of κ_E. Hence, $\kappa_E(x) = 0$. Moreover, the intersection of intervals $[t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma]$ and $[x-\Theta_\gamma,x]$ (on which the integral in (3.22) is computed) is interval $[\Theta_{\max}-\Theta_\gamma, x-\Theta_\gamma]$. Then, relations (3.21), (3.22) yield the following equality:

$$\sum_{i=0}^{i=n}\frac{c_i}{i+1}\times\left[(\Theta_{\max}-\Theta_\gamma)^{i+1}-(x-\Theta_\gamma)^{i+1}\right] = 0. \tag{3.23}$$

The polynom making up the left hand side of equality (3.23) is zero on interval $(\Theta_{\max} - \Theta_\gamma, \Theta_{\max})$ which has an non empty interior. Consequently, coefficients $(c_i)_{0\leq i\leq n}$ are zero.

On the other hand, we show that coefficients $(c_i)_{0\leq i\leq n}$ are zero, if $\Theta_{\max}-\Theta_\gamma < t_{\mathtt{I}}+\Theta_\gamma$. In this case, for any x such that $\Theta_{\max}-\Theta_\gamma < t_{\mathtt{I}}+\Theta_\gamma < x < \Theta_{\max}$, $\kappa_E(x) = 0$ and the intersection of intervals $[t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma]$ and $[x-\Theta_\gamma,x]$ is interval $[\Theta_{\max}-\Theta_\gamma, x-\Theta_\gamma]$. Hence, we obtain that the polynom of the left hand side of relation (3.23) is zero on interval $(t_{\mathtt{I}}+\Theta_\gamma, \Theta_{\max})$ which has an non empty interior. Consequently, coefficients $(c_i)_{0\leq i\leq n}$ are zero.

We showed that coefficients $(c_i)_{0\leq i\leq n}$ are zero in both cases $t_{\mathtt{I}}+\Theta_\gamma \leq \Theta_{\max}-\Theta_\gamma$ and $\Theta_{\max} - \Theta_\gamma < t_{\mathtt{I}}+\Theta_\gamma$. From this, we can deduce that (3.19) is true for any integer n.

Hence by density of polynoms in $\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])$ the proof of the lemma is achieved. $\square$

Lemma 3.6. *Assuming* (3.15)*, that implies:*

$$\int_0^{\Theta_\gamma} y\gamma(y)\, dy = \frac{\Theta_\gamma^2}{2} \neq 0, \tag{3.24}$$

is achieved and if function $\mathscr{L}[\kappa_E]$ given by relation (3.3) *satisfies:*

$$\int_{t_{\mathtt{I}}}^{\Theta_{\max}}\left(\mathscr{L}[\kappa_E](y) + \alpha\int_{t_{\mathtt{I}}}^{y}\mathscr{L}[\kappa_E](s)\mathrm{e}^{\alpha(y-s)}\, ds\right) dy = 0, \tag{3.25}$$

then, $\mathscr{F}(\kappa_E) \in \mathbb{L}^\infty(\mathbb{R})$ *and is such that*

$$\lim_{\xi \to 0} \mathscr{F}(\kappa_E)(\xi) = -\frac{2}{\Theta_\gamma^2} \int_{t_{\mathtt{I}}}^{\Theta_{\max}} y \times \left(\mathscr{L}[\kappa_E](y) + \alpha \int_{t_{\mathtt{I}}}^{y} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(y-s)} \, ds \right) dy. \tag{3.26}$$

If $\mathscr{L}[\kappa_E]$ *does not satisfy the equality in relation* (3.25)*, then,* $\mathscr{F}(\kappa_E)$ *has an infinite limit in* 0*.*

Proof. As $\mathscr{L}[\kappa_E] \in \mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$, we get:

$$\mathscr{L}[\kappa_E] + \alpha \int_{t_{\mathtt{I}}}^{\bullet} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(\bullet - s)} \, ds \in \mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}]). \tag{3.27}$$

Then, using an order 1 Taylor expansion of $e^{-iy\xi}$, we obtain the following expansion of function $\mathscr{F}(\mathscr{L}[\kappa_E] + \alpha \int_{t_{\mathtt{I}}}^{\bullet} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(\bullet - s)} \, ds)$:

(3.28)

$$\mathscr{F}\left(\mathscr{L}[\kappa_E] + \alpha \int_{t_{\mathtt{I}}}^{\bullet} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(\bullet - s)} \, ds \right)(\xi) = \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \left(\mathscr{L}[\kappa_E](y) + \alpha \int_{t_{\mathtt{I}}}^{y} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(y-s)} \, ds \right) dy$$
$$- i\xi \int_{t_{\mathtt{I}}}^{\Theta_{\max}} y \times \left(\mathscr{L}[\kappa_E](y) + \alpha \int_{t_{\mathtt{I}}}^{y} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(y-s)} \, ds \right) dy + O(\xi^2).$$

Since operator $\mathscr{L}$ satisfies equality in relation (3.25), relation (3.28) gives:

$$\begin{aligned} &\mathscr{F}\left(\mathscr{L}[\kappa_E] + \alpha \int_{t_{\mathtt{I}}}^{\bullet} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(\bullet - s)} \, ds \right)(\xi) \\ &\qquad = -i\xi \int_{t_{\mathtt{I}}}^{\Theta_{\max}} y \times \left(\mathscr{L}[\kappa_E](y) + \alpha \int_{t_{\mathtt{I}}}^{y} \mathscr{L}[\kappa_E](s) \mathrm{e}^{\alpha(y-s)} \, ds \right) dy + O(\xi^2). \end{aligned} \tag{3.29}$$

According to (3.15), function $1 - \mathscr{F}(\gamma)$ is Taylor expanded in 0 until the order 1 to obtain:

$$1 - \mathscr{F}(\gamma)(\xi) = i\xi \int_0^{\Theta_\gamma} y\gamma(y) \, dy + O(\xi^2) = i\xi \frac{\Theta_\gamma^2}{2} + O(\xi^2). \tag{3.30}$$

According to relations (3.6), (3.29) and (3.30), we get equality (3.26).

Moreover, according to relation (3.6), $\mathscr{F}(\kappa_E)$ is a finite quantity outside of 0. Hence, it is concluded that $\mathscr{F}(\kappa_E)$ is in $\mathbb{L}^\infty(\mathbb{R})$.

On other hand, if equality in relation (3.25) is not satisfied, then, according to relations (3.6), (3.28) and (3.30), $\mathscr{F}(\kappa_E)$ has an infinite limit at 0.

From this, the proof of the lemma is achieved. □

Lemma 3.7. *Under assumptions* (3.15) *and* (3.25)*,*

$$\left(\frac{1}{1-\mathscr{F}(\gamma)}\right)_{|(-\infty,-\frac{2}{\Theta_\gamma})\cup(\frac{2}{\Theta_\gamma},+\infty)} \in \mathbb{L}^\infty(\mathbb{R}), \tag{3.31}$$

and $\mathscr{F}(\kappa_E) \in \mathbb{L}^2(\mathbb{R})$.

In Lemma 3.7, $\left(\frac{1}{1-\mathscr{F}(\gamma)}\right)_{|(-\infty,-\frac{2}{\Theta_\gamma})\cup(\frac{2}{\Theta_\gamma},+\infty)}$ stands for the restriction of $\left(\frac{1}{1-\mathscr{F}(\gamma)}\right)$ to set $(-\infty,-\frac{2}{\Theta_\gamma})\cup(\frac{2}{\Theta_\gamma},+\infty)$.

Proof. As a consequence of assumptions (3.15), equality (3.24) holds and consequently

$$\left|\frac{1}{1-\mathscr{F}(\gamma)(\xi)}\right|^2 = \frac{(\xi\Theta_\gamma)^2}{(\xi\Theta_\gamma-\sin(\xi\Theta_\gamma))^2+(\cos(\xi\Theta_\gamma)-1)^2} \leq \left(\frac{\xi\Theta_\gamma}{\xi\Theta_\gamma-\sin(\xi\Theta_\gamma)}\right)^2. \tag{3.32}$$

Hence, if $|\xi| \geq 2/\Theta_\gamma$

$$\left|\frac{1}{1-\mathscr{F}(\gamma)(\xi)}\right| \leq 2, \tag{3.33}$$

which implies property (3.31).

According to (3.27), we get:

$$\mathscr{F}\left(\mathscr{L}[\kappa_E]+\alpha\int_{t_{\mathtt{I}}}^{\bullet}\mathscr{L}[\kappa_E](s)\mathrm{e}^{\alpha(\bullet-s)}\,ds\right) \in \mathbb{L}^2(\mathbb{R}). \tag{3.34}$$

Consequently, according to relations (3.6) and (3.32)

$$\mathscr{F}(\kappa_E)_{|(-\infty,-\frac{2}{\Theta_\gamma})\cup(\frac{2}{\Theta_\gamma},+\infty)} \in \mathbb{L}^2(\mathbb{R}) \tag{3.35}$$

Finally, from Lemma 3.6, we know $\mathscr{F}(\kappa_E) \in \mathbb{L}^\infty(\mathbb{R})$, which gives, using (3.35) that $\mathscr{F}(\kappa_E) \in \mathbb{L}^2(\mathbb{R})$, ending the proof of the Lemma □

Proof of Theorem 3.4. Under the assumptions (3.15) and (3.16) of Theorem 3.4, and using equality (3.5), that gives $\mathscr{L}[\kappa_E](t) = \sigma(t) - \mathscr{D}[\rho^{\mathtt{I}}_{\mathscr{K}}](t)$, we can apply Lemma 3.7 to deduce that formula (3.13) of Theorem 3.3 giving κ_E makes sense and yields κ_E in $\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])$. Besides, from Lemma 3.5, we obtain the uniqueness of κ_E, ending the proof of the theorem. □

4. Other financial models

4.1. **Model with variable rate.** The model we set out in Subsection 2.2 considered a constant rate. Nevertheless, if we consider in it a function α that depends on t, the model becomes a financial model with variable rate. The only modification to make is to enrich (2.10) by writing

$$\rho_{\mathscr{I}}(t) = \alpha(t)\,\mathscr{K}_{RD}(t). \tag{4.1}$$

Once this enrichment is done, the rest of Subsection 2.2 and Subsection 2.3 can be led with essentially no modification and the explanations remain true.
However, the mathematical results of section 3 are not true anymore in this context, and clearly using a variable rate makes the question of expressing Loan Density κ_E in terms of Algebraic Spending Density σ harder.

4.2. **Model with constant rate set at instants of borrowing.** In the real world, for loans that are at fixed rates, the rates are not the same all along the time. In fact, the fixed rate associated with a borrowed amount is set at the time when the amount is borrowed. The goal of this subsection is to enrich the model built above in order to account for this reality.

For this, we consider that all measures involved are density measures (we will see at the end of the subsection trails to generalize the resulting model to mesures that are not density measures). The main new thing that we need to introduce is quantity k_{RD} that details Current Debt Field $\mathscr{K}_{RD}$. Quantity $k_{RD}(t,s)$, that is so called Current Debt Density, depends on two variables (t and s) and is a time density with respect to variable s. It is defined as the density of Current Debt at time t which is associated to the amount borrowed at time s. In other words, the capital amount still to be paid at time t associated with amounts borrowed between two instants s_1 and s_2 is, by definition

$$\int_{s_1}^{s_2} k_{RD}(t,s)\,ds. \tag{4.2}$$

Current Debt Density k_{RD} is related to Loan Density κ_E using Repayment Pattern γ by the following Ordinary Differential Equation:

$$\frac{dk_{RD}(t,s)}{dt} = -\gamma(t-s)\kappa_E(s), \tag{4.3}$$

with initial condition $k_{RD}(s,s) = \kappa_E(s)$ that expresses that The Current Debt at time s which is related to the borrowed amount at time s is the borrowed amount at time s.
To justify the pertinence of equation (4.3), we notice that, in view of the definition of Repayment Pattern γ, $\gamma(t-s)$ is the repayment at time t of an amount 1 which is borrowed at time s. Hence, for a given amount A borrowed at time s, $\gamma(t-s)A$ is the repayment at time t of amount A. Consequently, $\gamma(t-s)\kappa_E(s)$ is the density (with respect to s) of what is repaid at time t associated with what is borrowed at time s. As this repayment density is nothing but the decreasing rate of $k_{RD}(t,s)$, (4.3) seems consistent.
The solution of this differential equation is expressed as:

$$k_{RD}(t,s) = \kappa_E(s) - \int_s^t \gamma(\sigma - s)\kappa_E(s)\, d\sigma. \tag{4.4}$$

The definition of Current Debt Density k_{RD} is consistent with the definition of Current Debt $\mathscr{K}_{RD}$ which is given in relation (2.19). Indeed, Current Debt $\mathscr{K}_{RD}$ can be expressed in terms of k_{RD}: it is the integral over all instants since $t_{\mathtt{I}}$ of what has to be repaid associated with what is borrowed at those instants plus the Current Debt at time $t_{\mathtt{I}}$ minus what has been repaid from this initial Current Debt:

$$\mathscr{K}_{RD}(t) = \int_{t_{\mathtt{I}}}^t k_{RD}(t,s)\, ds + \mathscr{K}_{RD}(t_{\mathtt{I}}) - \int_{t_{\mathtt{I}}}^t \rho_{\mathscr{K}}^{\mathtt{I}}(s)\, ds. \tag{4.5}$$

Using equalities (2.11) and (4.4) in this last formula, we obtain

$$\begin{aligned} \mathscr{K}_{RD}(t) &= \int_{t_{\mathtt{I}}}^t k_{RD}(t,s)\, ds + \int_t^{+\infty} \rho_{\mathscr{K}}^{\mathtt{I}}(s)\, ds \\ &= \int_{t_{\mathtt{I}}}^t \kappa_E(s)\, ds - \int_{t_{\mathtt{I}}}^t \left(\int_s^t \gamma(\sigma - s)\kappa_E(s)\, d\sigma \right) ds + \int_t^{+\infty} \rho_{\mathscr{K}}^{\mathtt{I}}(s)\, ds. \end{aligned} \tag{4.6}$$

Since

$$\int_{t_{\mathtt{I}}}^{t}\left(\int_{s}^{t}\gamma(\sigma-s)\kappa_E(s)\,d\sigma\right)ds=\int_{t_{\mathtt{I}}}^{t}\left(\int_{t_{\mathtt{I}}}^{\sigma}\gamma(\sigma-s)\kappa_E(s)\,ds\right)d\sigma, \tag{4.7}$$

(4.6) yields

$$\mathscr{K}_{RD}(t)=\int_{t_{\mathtt{I}}}^{t}\kappa_E(s)\,ds-\int_{t_{\mathtt{I}}}^{t}\left(\int_{t_{\mathtt{I}}}^{\sigma}\gamma(\sigma-s)\kappa_E(s)\,ds\right)d\sigma+\int_{t}^{+\infty}\rho_{\mathscr{K}}^{\mathtt{I}}(s)\,ds. \tag{4.8}$$

Equality (4.8) has to be compared to (2.19), which in the context of density measures and using (2.9) to express $\rho_{\mathscr{K}}$ gives

$$\mathscr{K}_{RD}(t)=\int_{t_{\mathtt{I}}}^{t}\kappa_E(t)\,dt-\int_{t_{\mathtt{I}}}^{t}\left(\int_{-\infty}^{+\infty}\kappa_E(s)\gamma(\sigma-s)\,ds\right)d\sigma+\int_{t}^{+\infty}\rho_{\mathscr{K}}^{\mathtt{I}}(t)\,dt. \tag{4.9}$$

As γ is 0 on $\mathbb{R}^-$, $\gamma(\sigma-s)$ is 0 for s larger than σ. On another hand $\kappa_E(s)$ is 0 for s smaller than $t_{\mathtt{I}}$. Hence (4.9) finally read

$$\mathscr{K}_{RD}(t)=\int_{t_{\mathtt{I}}}^{t}\kappa_E(t)\,dt-\int_{t_{\mathtt{I}}}^{t}\left(\int_{t_{\mathtt{I}}}^{\sigma}\kappa_E(s)\gamma(\sigma-s)\,ds\right)d\sigma+\int_{t}^{+\infty}\rho_{\mathscr{K}}^{\mathtt{I}}(t)\,dt, \tag{4.10}$$

which is exactly (4.8).

The other introduced thing is Density $r_{\mathscr{I}}(t,s)$. It is a time density with respect to both variables s and t. It is the Interest Payment Density at time t which is associated with the borrowed amount at time s. It is related to the Current Debt Density k_{RD} by a proportionality relation:

$$r_{\mathscr{I}}(t,s)=\alpha_{\mathrm{SAB}}(s)\,k_{RD}(t,s), \tag{4.11}$$

where, $\alpha_{\mathrm{SAB}}(s)$ is the value of the rate at the instant s at which the amount is borrowed (SAB is for Set At Borrowed time). The expression of Borrowed Time Related Interest Payment Density $r_{\mathscr{I}}$ may be given in terms of Loan Density κ_E as follows:

$$r_{\mathscr{I}}(t,s)=\alpha(s)\kappa_E(s)-\alpha_{\mathrm{SAB}}(s)\int_{s}^{t}\gamma(\sigma-s)\kappa_E(s)\,d\sigma. \tag{4.12}$$

By integration over variable s (which describes the borrowed time), from $r_{\mathscr{I}}(t,s)$, Interest Payment Density $\rho_{\mathscr{I}}$ can be defined as follows:

$$\rho_{\mathscr{I}}(t)=\int_{t_{\mathtt{I}}}^{t}r_{\mathscr{I}}(t,s)\,ds+\rho_{\mathscr{I}}^{\mathtt{I}}(t), \tag{4.13}$$

where $\rho^{\mathtt{I}}_{\mathscr{I}}(t)$ is a third new thing: it is the Interest Payment Density which is related to the Current Debt Field at the initial instant $t_{\mathtt{I}}$. It is considered as given (as $\rho^{\mathtt{I}}_{\mathscr{K}}$ and $\mathscr{K}_{RD}(t_{\mathtt{I}})$ are). From (4.12) and (4.13), Interest Payment Density $\rho_{\mathscr{I}}$ reads

$$\rho_{\mathscr{I}}(t) = \int_{t_{\mathtt{I}}}^{t} \alpha_{\mathrm{SAB}}(s)\kappa_E(s)\,ds - \int_{t_{\mathtt{I}}}^{t} \alpha_{\mathrm{SAB}}(s)\left(\int_{s}^{t} \gamma(\sigma - s)\kappa_E(s)\,d\sigma\right)ds + \rho^{\mathtt{I}}_{\mathscr{I}}(t). \tag{4.14}$$

Expression (4.14) and consequently definition of Borrowed Time Related Interest Payment Density $r_{\mathscr{I}}$ are consistent with the expression of Interest Payment Density $\rho_{\mathscr{I}}$ given in relation (2.20). Indeed, if the rate α_{SAB} is fixed with worth α, (4.14) writes

$$\rho_{\mathscr{I}}(t) = \alpha \int_{t_{\mathtt{I}}}^{t} \kappa_E(s)\,ds - \alpha \int_{t_{\mathtt{I}}}^{t} \left(\int_{s}^{t} \gamma(\sigma - s)\kappa_E(s)\,d\sigma\right)ds + \rho^{\mathtt{I}}_{\mathscr{I}}(t), \tag{4.15}$$

and is exactly (2.20), that when written in the context of density measures reads

$$\rho_{\mathscr{I}}(t) = \alpha \int_{t_{\mathtt{I}}}^{t} \kappa_E(s)\,ds - \alpha \int_{t_{\mathtt{I}}}^{t} \left(\int_{s}^{t} \gamma(\sigma - s)\kappa_E(s)\,d\sigma\right)ds + \alpha \int_{t}^{+\infty} \rho^{\mathtt{I}}_{\mathscr{K}}(s)\,ds, \tag{4.16}$$

with the following definition of $\rho^{\mathtt{I}}_{\mathscr{I}}$

$$\rho^{\mathtt{I}}_{\mathscr{I}}(t) = \alpha \int_{t}^{+\infty} \rho^{\mathtt{I}}_{\mathscr{K}}(s)\,ds. \tag{4.17}$$

Exchanging the role of s and σ in (4.14), it is possible to write expression of Interest Payments Density $\rho_{\mathscr{I}}$ in the next form:

$$\rho_{\mathscr{I}}(t) = \int_{t_{\mathtt{I}}}^{t} \alpha_{\mathrm{SAB}}(s)\kappa_E(s)\,ds - \int_{t_{\mathtt{I}}}^{t} \left(\int_{t_{\mathtt{I}}}^{\sigma} \alpha_{\mathrm{SAB}}(s)\gamma(\sigma - s)\kappa_E(s)\,ds\right)d\sigma + \rho^{\mathtt{I}}_{\mathscr{I}}(t). \tag{4.18}$$

Remark 4.1. *This last form gives a trail to generalize the model to measures that are not density measures. Indeed, it brings to think that it is possible to find a subspace of measure space $\mathscr{M}([t_{\mathtt{I}},\Theta_{\max}])$ that is the dual of a space of functions containing piecewise continuous functions such that* (4.18) *can be generalized as*

$$\rho_{\mathscr{I}}(t) = \left\langle \tilde{\kappa}_E, \alpha_{\mathrm{SAB}|[t_{\mathtt{I}},t[} \right\rangle - \left\langle \tilde{\gamma} \star (\alpha_{\mathrm{SAB}}\tilde{\kappa}_E)_{|[t_{\mathtt{I}},\sigma[}, \mathbb{1}_{[t_{\mathtt{I}},t[} \right\rangle + \rho^{\mathtt{I}}_{\mathscr{I}}(t), \tag{4.19}$$

where $\tilde{\kappa}_E$ is an element of this subspace of $\mathscr{M}([t_{\mathtt{I}},\Theta_{\max}])$ and where $\langle .,. \rangle$ stands for the duality bracket of this space with the space it is the dual of.

4.3. **Model with variable rate set at instants of borrowing.** We can generalize the model explained in subsection 4.2 by considering that rate α_{SAB} which is set at borrowed time is variable. For this, the only thing that needs to be done is considering that $\alpha_{\mathrm{SAB}}(s,t)$ depends on two variables (s and t). By definition $\alpha_{\mathrm{SAB}}(s,t)$ is the rate at time t that applies to amounts that were borrowed at time s. All formula in subsection 4.2 can be rewritten adding this enrichment. For instants, (4.18) is rewritten as:

$$\rho_{\mathscr{I}}(t) = \int_{t_{\mathtt{I}}}^{t} \alpha_{\mathrm{SAB}}(s,t)\kappa_E(s)\,ds - \int_{t_{\mathtt{I}}}^{t}\left(\int_{t_{\mathtt{I}}}^{\sigma} \alpha_{\mathrm{SAB}}(s,t)\gamma(\sigma-s)\kappa_E(s)\,ds\right)d\sigma + \rho_{\mathscr{I}}^{\mathtt{I}}(t). \tag{4.20}$$

4.4. **Model with varying Repayment Pattern.** We now enrich again our model to take into account variable Repayment Pattern γ. For this, we consider that $\gamma(s,t)$ depends on two variables. The first variable s is the borrowed instant and the second variable t is the current time. In order to ensure that amounts borrowed at any time s are repaid exactly, $\gamma(s,t)$ needs to satisfy the following variant of (2.8):

$$\int_{-\infty}^{+\infty} \gamma(s,\sigma)\,d\sigma = 1. \tag{4.21}$$

Generalizing (2.9), Loan Density κ_E and Repayment Density $\rho_{\mathscr{K}}$ are connected by:

$$\rho_{\mathscr{K}}(t) = \int_{-\infty}^{+\infty} \gamma(s,t-s)\kappa_E(s)\,ds. \tag{4.22}$$

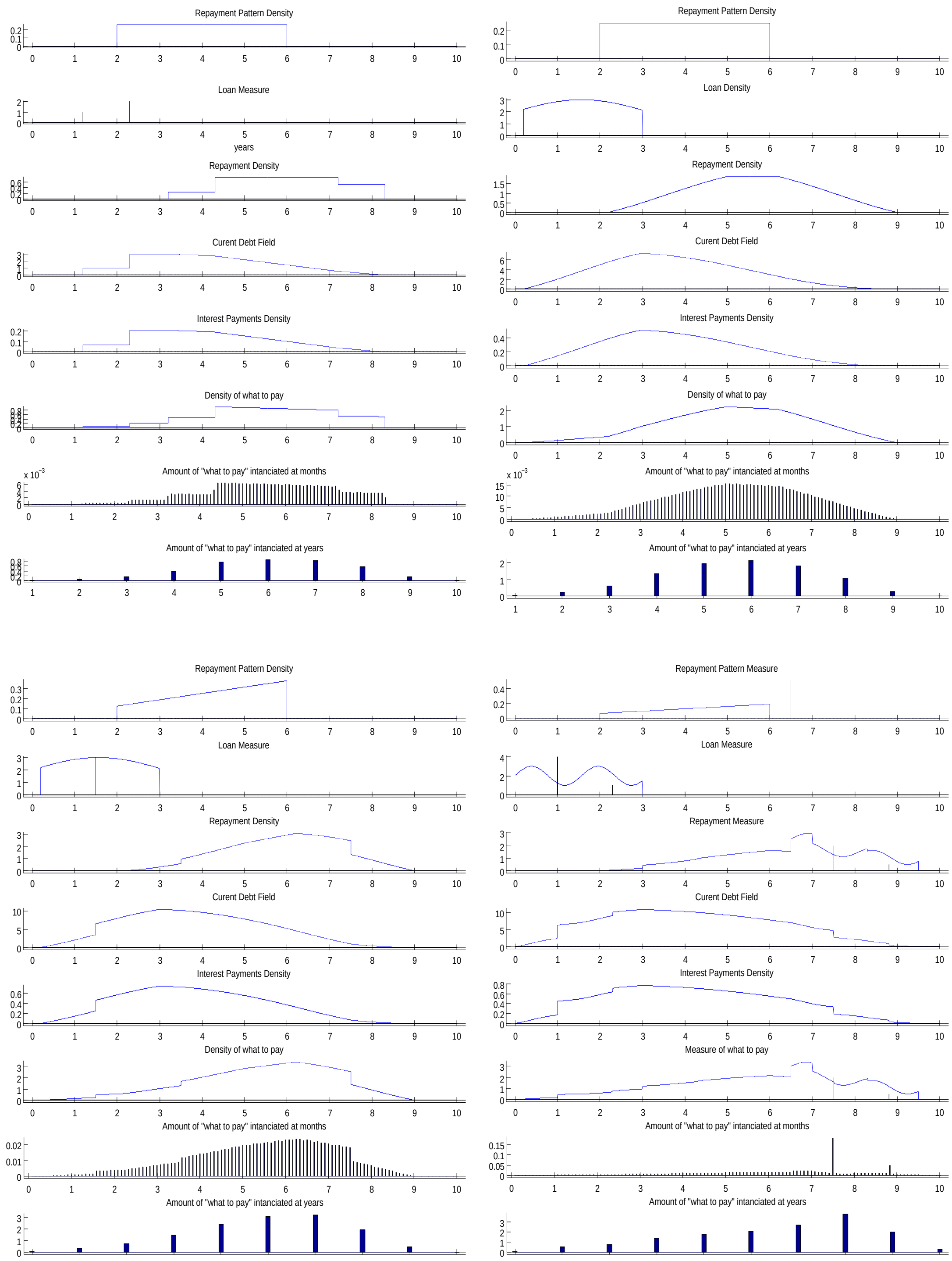

FIGURE 3. In each of the four pictures, $\tilde{\gamma}$ is represented in the top diagram; $\tilde{\kappa}_E$ in the second diagram from the top, $\tilde{\rho}_{\mathscr{K}}$, which is computed with relation (2.7), in the third one; $\mathscr{K}_{RD}$ which is computed according to relation (2.19) in the fourth one; $\rho_{\mathscr{I}}$, calculated according to relation (2.10), in the fifth; the sixth diagram gives the density of what is to paid. The two last diagrams are respectively the amounts of what to pay over monthly and yearly period.

Thanks to property (4.21), we have:

$$\int_{-\infty}^{+\infty} \rho_{\mathscr{K}}(t)\,dt = \int_{-\infty}^{+\infty} \kappa_E(t)\,dt. \tag{4.23}$$

The variation, with respect to time t, of the Current Debt k_{RD} which is associated to the amount borrowed at time s is equal to the product of Repayment Pattern $\gamma(s,t-s)$ by Loan Density κ_E at time s. Hence, we obtain the following generalization of relation (4.3):

$$\frac{dk_{RD}(t,s)}{dt} = -\gamma(s,t-s)\kappa_E(s). \tag{4.24}$$

Besides generalizing relation (4.18), Interest Payments Density $\rho_{\mathscr{I}}$ can be given as follows:

$$\rho_{\mathscr{I}}(t) = \int_{t_{\mathtt{I}}}^{t} \alpha_{\mathrm{SAB}}(s,t)\kappa_E(s)\,ds - \int_{t_{\mathtt{I}}}^{t}\int_{t_{\mathtt{I}}}^{\sigma} \alpha_{\mathrm{SAB}}(s,t)\gamma(s,\sigma-s)\kappa_E(s)\,ds\,d\sigma + \rho_{\mathscr{I}}^{\mathtt{I}}(t). \tag{4.25}$$

5. Model using

In this section, we show on simplified examples how the previously built models can be used by an organization to prepare its future financial plans.

5.1. **Forecasts.** In this subsection, we show how the model can be used by a local community, or any organization, to forecast the consequences of a loan in the future.
In a first place, we briefly recall the comments on the pictures of Figure 2 with this slant. In the top left picture, we see in the bottom digram, the reimbursement of an amount borrowed at once (see the middle diagram), according to Repayment Pattern of the top diagram. In the top right picture, the bottom digram shows the reimbursement of an amount borrowed in two (see the middle diagram), and in the bottom left picture, the bottom digram shows the reimbursement of an amount borrowed with a more complex scheme. The bottom digram of the bottom right picture shows the reimbursement density associated to the loan density shown in the middle diagram, according to the Repayment Pattern shown in the top diagram.
Of course, we could comment similarly on pictures of Figure 1 where various Repayment Patterns (top diagrams) and Loan Measures (middle diagrams) yield Repay Measures (bottom diagrams).

Now we turn to richer simulations. Figure 3 shows four simulations in four pictures. Each picture shows eight diagrams. For every simulation, initial time $t_{\mathtt{I}} = 0$, Current Debt at initial time $\mathscr{K}_{RD}(t_{\mathtt{I}}) = 0$ and Initial Debt Repayment Plan $\tilde{\rho}^{\mathtt{I}}_{\mathscr{K}} = 0$. If we stand in the context of an organization that needs to know its finance state in futur, the two last quantities describe its finance state at the beginning of the considered period. And, loan rate α and Repayment Pattern $\tilde{\gamma}$ is a translation of the loan contract that the organization has with it financial institution. In every presented simulation loan rate $\alpha = 0.07$.

The diagram at the top of each picture is the Repayment Pattern $\tilde{\gamma}$ associated to the simulation. The second diagram is the Loan Measure $\tilde{\kappa}_E$. This measure can be interpreted as the way the organization is going to borrow and how much it will borrow. Once those quantities laid down, thanks to our model, Repayment Measure $\tilde{\rho}_{\mathscr{K}}$ is obtained from $\tilde{\gamma}$ and $\tilde{\kappa}_E$ by (2.7). The quantities drawn in the fourth and fifth diagrams are respectively the Current Debt Field $\mathscr{K}_{RD}$, given by equality (2.19), and the Interest Payments Density $\rho_{\mathscr{I}}$ defined by (2.10). The sixth diagram is the Measure of what has to be paid to the financial institution defined by $\tilde{\rho}_{\mathscr{K}} + \tilde{\rho}_{\mathscr{I}}$. This is a forecast of what the organization will pay in future. From measure $\tilde{\rho}_{\mathscr{K}} + \tilde{\rho}_{\mathscr{I}}$, the amounts of what to pay over monthly and yearly period can deduced by integration. This can refine the forecast. Those amounts are represented in the seventh and eighth diagram.

Those simulations illustrate the capability of our model to help the setting out of finance forecast. Moreover, they illustrate the capability of our approach to overcome a problem occurring with the presently used tools (see page 2): With the continuous-in-time approach, the computations are led without any question concerning time period at which things are going to be observed. Indeed, those questions arise only at the very end of the process to compute monthly or annual installments, or any quantity which is related to a period of time.

Another capability which is illustrated by those examples is that our approach is flexible and can incorporate complexity, concentrated and continuous objects in a same simulation that makes sense. Their is no limitation in the choice of the Repayment Pattern and of the Loan Measure. Hence, we can account for a wide variety real situations.

Let us go into more detailed explanations on what can be seen in pictures of Figure 3.

The top left picture shows a Repayment Pattern Density γ which is a piecewise function that is equal to 0.25 between instants 2 and 6 and to 0 elsewhere. On the second diagram is drawn Loan Measure $\tilde{\kappa}_E$ which is a sum of two Dirac masses. The first Dirac mass is of mass 1 at time 1.25 and the second one is of mass 2 at time 2.25.

The Repayment Pattern Density in the top right picture is the same as before and the Loan Density is a sinusoidal function over time period $[0.2, 3]$ extended by 0.

The top diagram of the bottom left picture is Repayment Pattern Density γ which is an affine function over an interval extended by 0. In this simulation, Loan Measure $\tilde{\kappa}_E$ is the one of the top right picture, to which is added a Dirac mass that has mass 3 at time 1.5.

In the bottom right picture, the Repayment Pattern is the previous one plus a Dirac mass of mass 0.5 located at time 6.5. The loan Measure (see second diagram) is the sum of a sinusoidal function and of Dirac mass with mass 4 at time 1.

5.2. **Financial strategy.** For an organization, setting out a strategy needs to go beyond doing forecasts. For financial strategy, it needs to build a budget project, then to estimate its impact on the future finance health of the organization and if it is not satisfactory, to rebuild a new budget project.

In this subsection we will show how to use our continuous-in-time model for those kind of purposes.

We first comment on Figure 4. The first diagram is the Repayment Pattern. It translates that the organization has to repay borrowed amounts constantly just after borrowing. Then, on a second period, the repayment decreases, and then stops. During the fourth period the repayment increases strongly.

The second diagram shows the Loan Measure. It is a translation of how much the organization plan to borrow and according to what plan it is going to do it. In the third diagram is shown the Repayment Measure. It can be interpreted as an impact of the choice of the Repayment Pattern and of the Loan Measure. Now we can imagine that decision makers of the organization do not find this Repayment Measure convenient and that they would prefer another one: the one which is given in the fourth diagram. Once this targeted Repayment Measure is chosen, using formula (2.7) (or (2.9)), which reads using Fourier Transform $\mathscr{F}(\tilde{\gamma})\mathscr{F}(\tilde{\kappa}_E) = \mathscr{F}(\tilde{\rho}_{\mathscr{K}})$, allows us to compute a Loan Measure that will give it. This Loan Measure is given in the fifth diagram. This resulting Loan Measure has some problems: It is not non-negative and its support is two large. Hence we rid those problems by suppressing its negative part and reducing its support, leading the loan measure shown in the sixth diagram. In order to be sure that this Loan Measure fits well the decision makers' will, its associated Repayment Measure is computed and given in the last diagram.

This example illustrates well the capability of our model to be used in a strategy elaboration process. This capability is entirely linked with its continuous-in-time nature. Indeed, this nature brings to handle objects that are defined over a time interval of interest and mathematical operators that link them together. Consequently, strategy-related questionings reduce to question of inverting those operators. And, even if most of those operators are not invertible, solutions of the posed inversion problem can be brought (as it is the case in this example and consistently with the contents of section 3).

. The example of Figure 5 is richer than the previous one and involves the same Repayment Pattern and initial Loan Measure. Yet, it involves moreover a Current Spending Measure $\tilde{\sigma}_g$, that we chose to be zero, which was not considered in the example related with Figure 4 and e. Using formula (3.2), the Algebraic Spending Measure $\tilde{\sigma}$ can be computed. Consequently the Measure of Isolated Spending $\tilde{\beta}$, that models how much money can be put into a project led by the organization, can be deduced according to formula (2.13), which is same as $\tilde{\sigma}$.

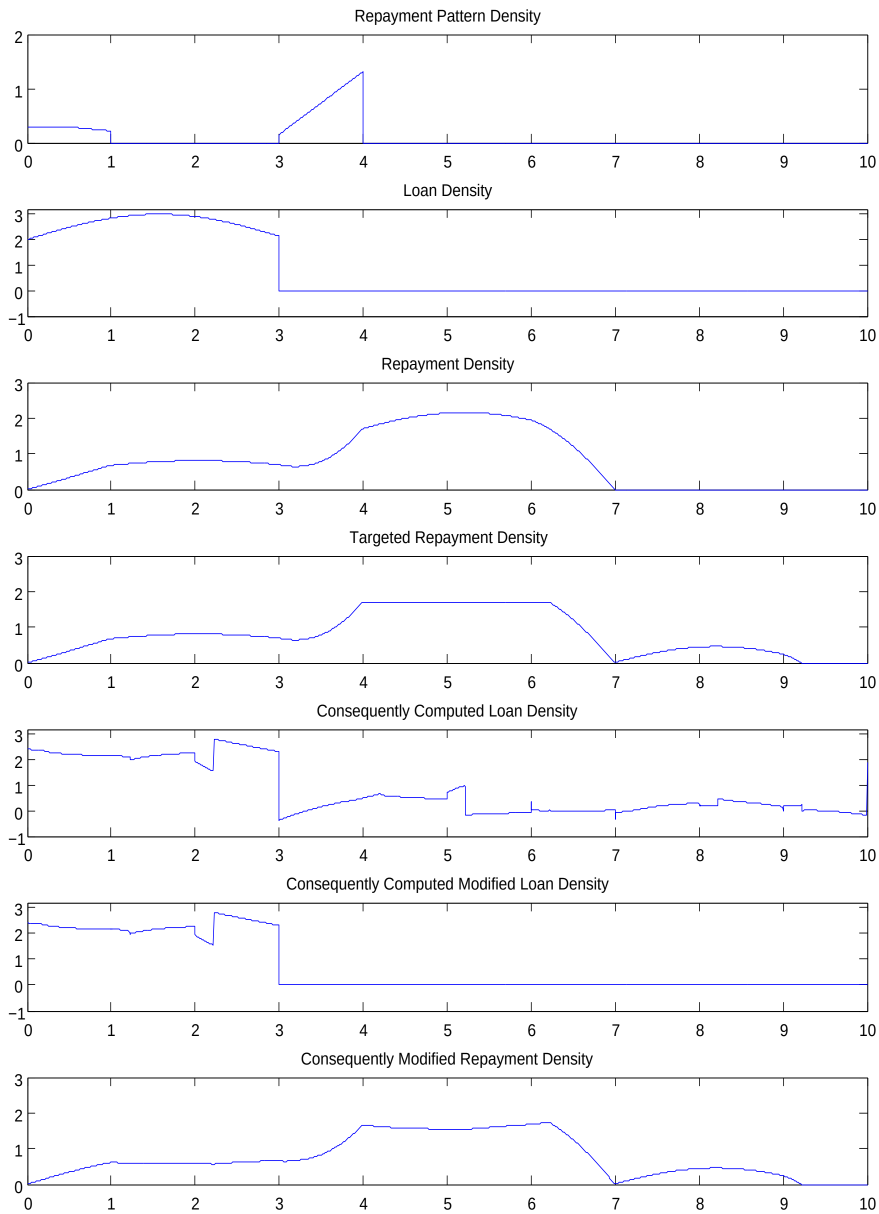

FIGURE 4. $\tilde{\gamma}$ is represented in the top diagram, $\tilde{\kappa}_E$ in the second diagram from the top, $\tilde{\rho}_{\mathscr{K}}$ in the third diagram from the top which is computed with relation (2.7), the fourth diagram from the top is targeted Repayment Density which is supposed to be given, an new Loan Density is computed in the fifth diagram from the top, this density is modified to generate also a new Loan Density in the sixth diagram from the top, the bottom diagram is a consequently modified Repayment Density $\tilde{\rho}_{\mathscr{K}}$.

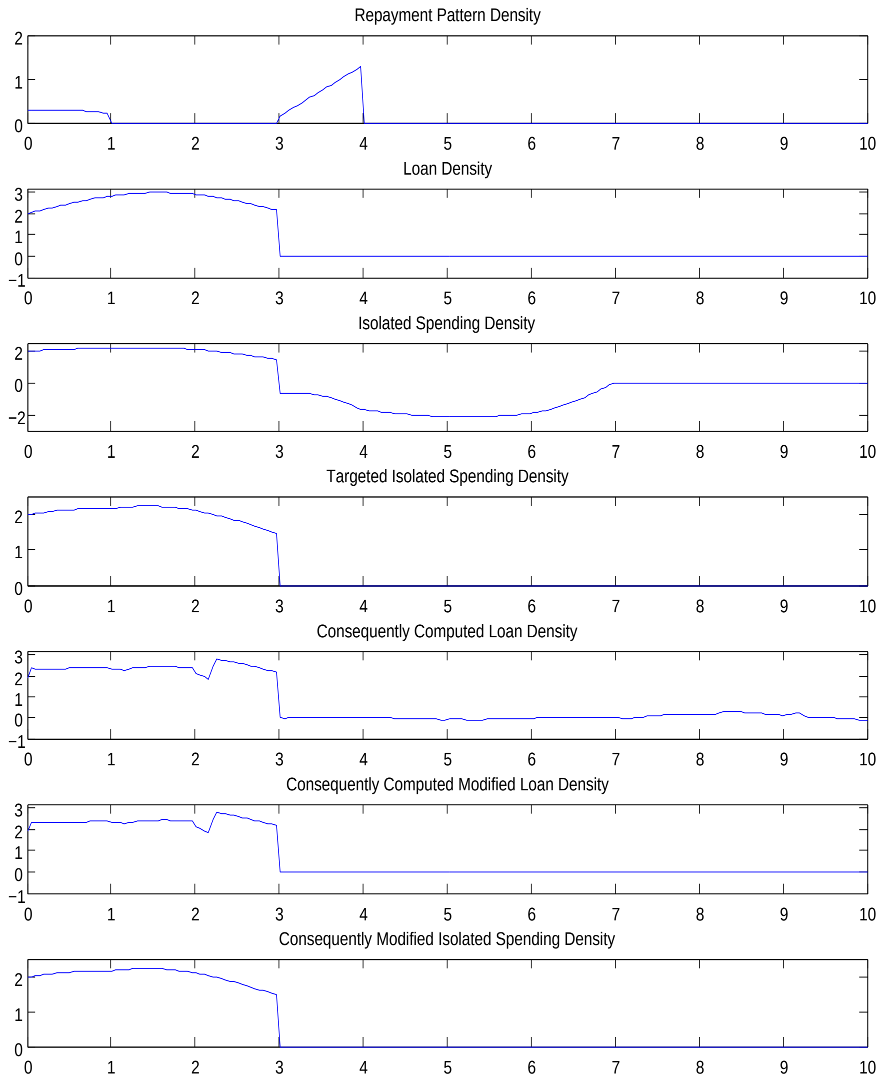

FIGURE 5. $\tilde{\gamma}$ is represented in the top diagram, $\tilde{\kappa}_E$ in the second diagram from the top, $\tilde{\beta}$ in the third diagram from the top which is computed with relation (3.6) or (5.2), the fourth diagram from the top is targeted Isolated Spending Density, an new Loan Density is computed in the fifth diagram from the top, this density is modified to generate also a new Loan Density in the sixth diagram from the top, the bottom diagram is a consequently modified Isolated Spending Density which is same as $\tilde{\beta}$.

The Isolated Spending Measure $\tilde{\beta}$ is drawn in the third diagram. Now, as in the previous example, we can imagine that Isolated Spending Measure $\tilde{\beta}$ does not fit the vision of some decision makers. Then it can be modified to get a more convenient one, drawn in the fourth diagram. Once the targeted Measure of Isolated Spending $\tilde{\beta}$ set, applying formula (3.6), setting Measure $\tilde{\beta}$ gives an associated Loan Measure (see the fifth diagram) that needs to be cleaned (see the sixth diagram). In order to be sure that this Loan Measure fits well the decision makers' objectives, its associated Isolated Spending Measure is computed and given in the last diagram.

This example illustrates on a more complicated case that our model can be used for strategy elaboration. This complication consists in giving Loan Measure $\tilde{\kappa}_E$ as function of Isolated Spending Measure $\tilde{\beta}$ under some assumptions. In order to make the simulations shown in Figure 5, $\tilde{\kappa}_E$ is computed in terms of $\tilde{\beta}$ following several steps. These steps are the followings, in this example where $\tilde{\sigma}$ and $\tilde{\beta}$ are equals. The first step is to compute $\mathscr{L}[\kappa_E]$ as a function of $\tilde{\beta}$ and $\mathscr{D}[\rho^{\mathtt{I}}_{\mathscr{K}}]$ using relation (3.5). The second step is to compute Fourier Transform of quantities $\mathscr{L}[\kappa_E]+\alpha\int_{t_{\mathtt{I}}}^{\bullet}\mathscr{L}[\kappa_E](s)\mathrm{e}^{\alpha(\bullet-s)}\,ds$ and γ. If Inverse Fourier Transform of quantity

$$\frac{\mathscr{F}(\mathscr{L}[\kappa_E]+\alpha\int_{t_{\mathtt{I}}}^{\bullet}\mathscr{L}[\kappa_E](s)\mathrm{e}^{\alpha(\bullet-s)}\,ds)}{(1-\mathscr{F}(\gamma))} \tag{5.1}$$

exists, then, $\tilde{\kappa}_E$ can be computed in term of $\tilde{\beta}$ with using relation (3.6) and it is given as follows:

$$\kappa_E=\mathscr{F}^{-1}\left(\frac{\mathscr{F}(\mathscr{L}[\kappa_E]+\alpha\int_{t_{\mathtt{I}}}^{\bullet}\mathscr{L}[\kappa_E](s)\mathrm{e}^{\alpha(\bullet-s)}\,ds)}{1-\mathscr{F}(\gamma)}\right). \tag{5.2}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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References

[1] M. Benaim and J.-Y. Le Boudec. A class of mean field interaction models for computer and communication systems. *Performance Evaluation*, 65(11-12):823–838, 2008.

[2] Emmanuel Frénod, Pierre Menard, and Mohamad Safa. Optimal control of a continuous-in-time financial model. *Mathematical Modelling and Numerical Analysis*, 2013.

[3] Emmanuel Frénod, Pierre Menard, and Mohamad Safa. Two optimization problems using a continuous-in-time financial model. *Journal of Industrial and Management Optimization*, 2014.

[4] Emmanuel Frénod and Mohamad Safa. Continuous-in-time financial model for public communities. pages 1–10, 2013.

[5] Nicolas Gast, Bruno Gaujal, and Jean-Yves Le Boudec. Mean field for markov decision processes: from discrete to continuous optimization. *Inria RR 7239*, 2011.

[6] Francois Golse. *On the Dynamics of Large Particle Systems in the Mean Field Limit*. Ecole Polytechnique, Centre de Mathématiques Laurent Schwartz, 91128 Palaiseau Cedex, France, 1992.

[7] Anne Hanselmann, Oliver C Schrempf, and Uwe D Hanebeck. Optimal parametric density estimation by minimizing an analytic distance measure. In *Information Fusion, 2007 10th International Conference on*, pages 1–8. IEEE, 2007.

[8] Vesa Klumpp and Uwe D Hanebeck. Direct fusion of dirac mixture densities using an efficient approximation in joint state space. In *Multisensor Fusion and Integration for Intelligent Systems, 2008. MFI 2008. IEEE International Conference on*, pages 168–174. IEEE, 2008.

[9] R.C. Merton. *Continuous-Time Finance*. Blackwell, 1st edition, 1992.

[10] S.M. Sundaresan. Continuous-time methods in finance: A review and an assessment. *Journal of Finance*, pages 1569–1622, 2000.