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FINITE DIFFERENCE APPROACH TO THE VALUATION AND PREPAYMENT STRATEGY OF MORTGAGES

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Abstract. This paper studies the value of mortgage contract by assuming the market interest rate follows the Cox-Ingersoll-Ross (CIR) model and correspondingly the borrower's optimal strategy to make prepayment. The problem is formulated as a partial differential equation with initial and boundary conditions imposed by the contract conditions. Finite difference approach is applied to solve (1) the optimal prepayment interest rate; (2) the value of the mortgage contract when prepayment is allowed. In addition, numerical solutions are verified with analytical asymptotic results for the small volatility scenario.

Keywords: Numerical method; Finite difference; Mortgage valuation; Mortgage prepayment.

2010 AMS Subject Classification: 91B25, 91B26, 35K10, 65M06.

1. Introduction

The valuation of mortgages is an intriguing problem in finance. When prepayment is allowed, as is a typical condition offered by lenders to attract home buyers, the borrower faces an optimal control problem in the filtration of market information [12, 20, 21]. Prepayment grants a borrower the right to choose any time to settle the loan balances all at once by taking

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advantage of the lower interest rate. The borrower can leverage on the choice of prepayment to optimize his portfolio so that a maximum possible return can be made in statistical sense given the knowledge of the underlying market interest rate. According to Feymann-Kac theorem (see [5, 14, 9], for instance), the value of such a contract is determined by a system of partial differential equations, with boundary and initial conditions imposed by the contract clauses such as the originating loan amount and monthly payment plans. Because of the important role played by the mortgage securities in real economy, there exists a considerable literature (see [2, 10, 15, 4], for example) dedicated to the topic. In recent development, a free boundary approach was applied to valuation of mortgages where the free boundary defines the threshold market interest rate below which it is optimal for the borrower to settle the mortgage balance ([17, 18]). The integral representation of the solutions to mortgage valuation problems as used in [18], for example, requires the existence of the corresponding Green's functions, making such solutions less flexible, in some sense, from financial practitioners' perspective. Classic binomial scheme for the problem is also possible. Examples of using binomial tree to solve the problem can be found in [8, 9]. But as it is commonly known, the binomial tree method is low in computing speed and rate of convergence [1, 7]. This work intends to solve the problem using finite difference approach. The feasibility of the approach applied for the valuation of options with early exercise features is suggested by [13].

We consider a mortgage contract model where the debtor pays m (dollars) per month with monthly interest rate r_0. At each time t, the outstanding mortgage balance, $M(t)$, is determined by the following ODE:

$$dM(t) = -m + r_0 M(t).$$

At the expiration time T, $M(T) = 0$. Then the ODE has a unique solution

$$M(t) = \frac{m}{r_0}(1 - e^{-m(T-t)}).$$

Assume that the debtor can terminate the contract by paying-off the outstanding balance all at once. Theoretically, there exists an optimal interest rate, denoted as $h(\tau)$, where $\tau = T - t$, for the borrower to make such a prepayment decision. The optimal prepayment boundary separates the domain of the spatial variable, which in our case is the interest rate, into two regions, namely, the continuation region within which the borrower continues paying the monthly installment and the early contract closing region within which the borrower settles the outstanding loan. The market interest rate in this paper is assumed to follow the CIR process [3], i.e.,

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dW(t), \tag{1}$$

where $W(t)$ is a Wiener process modeling the random market risk factor, α, β are positive constant parameters, and σ is a non-negative constant parameter.

To find the borrower's optimal strategy of prepayment, we set $V(r,t)$ as the expected value of the contract at time t with the current market interest rate r. As discussed in [17], the value $V(r,t)$ is a fair price at which a rational buyer would offer to take over the contract. Since the debtor can choose to prepay the loan at a lower interest rate, the range of $V(r,t)$ is from 0 to $M(t)$. Assuming that the debtor will choose to prepay the outstanding mortgage when the market interest rate is at $h(\tau)$ or below, we solve the problem by deriving the following system (see [17, 18]):

$$\begin{cases} \frac{1}{2}\sigma^2 r \frac{\partial^2 V}{\partial r^2} + (\alpha - \beta r)\frac{\partial V}{\partial r} - \frac{\partial V}{\partial \tau} = rV - m, & \text{for} \quad r > h(\tau), \tau > 0, \\ V(\tau, h(\tau)) = \frac{m}{r_0}\left(1 - e^{-r_0\tau}\right), & \text{for} \quad r \le h(\tau), \tau > 0, \\ V(\tau, r_{max}) = 0, & \tau \in [0, T], \\ V(0, r) = 0, & r > 0, \end{cases} \tag{2}$$

augmented with the 'smooth pasting' condition at the optimal exercise boundary $h(\tau)$

$$\frac{\partial V}{\partial r}(\tau, h(\tau)) = 0. \tag{3}$$

We let $\Omega = \{(\tau, r) | 0 \le \tau \le T, r_{min} \le r \le r_{max}\}$ to be the whole plane. We divide the r-axis into equally spaced nodes, distanced apart by δr, and the τ-axis equally distanced apart by $\delta\tau$. Then the total number of meshes in the r-axis is $J = [\frac{r_{max} - r_{min}}{\delta r}]$, while the total number of meshes in the τ-axis is $I = [\frac{T}{\delta\tau}]$.

We may convert the floating boundary in (2) to a fixed boundary by letting

$$y = \ln\frac{r}{h(\tau)}.$$

Thus, (2) can be transformed to

$$\begin{cases} \dfrac{\sigma^2}{2h(\tau)e^y}\dfrac{\partial^2 V}{\partial y^2} - \left[\dfrac{(\alpha-\beta h(\tau)e^y)}{h(\tau)e^y} + \dfrac{\sigma^2}{2h(\tau)e^y} + \dfrac{h'(\tau)}{h(\tau)}\right]\dfrac{\partial V}{\partial y} + \dfrac{\partial V}{\partial \tau} + h(\tau)e^y V = m, & \text{for} \quad y > 0 \\ V(\tau,0) = \dfrac{m}{r_0}\left(1-e^{-r_0\tau}\right), & \text{for} \quad y \le 0 \\ V(\tau, y_{max}) = 0, & \tau \in [0,T] \\ V(0,y) = 0, & \forall \quad y \in R. \end{cases} \tag{4}$$

Accordingly, the free boundary condition defined by (3) is transformed as

$$\frac{\partial V}{\partial y}(\tau,0) = 0. \tag{5}$$

In financial management, the rational debtor will compare the spot interest rate r_τ and the upper bounded interest rate $h(\tau)$. If r_τ is less than the theoretical boundary $h(\tau)$ at time τ, it is wise for the debtor to prepay. Otherwise, the debtor will wait until the spot interest rate reaches the boundary. Thus, the optimal boundary occurs at $r = h_\tau$, or equivalently $y = 0$.

2. The floating boundary

At $y = 0$, the PDE in (4) yields a lower boundary condition for the finite difference scheme proposed by this paper, i.e.,

$$\frac{\sigma^2}{2h(\tau)}\frac{\partial^2 V}{\partial y^2} = m\left(1-e^{-r_0\tau}\right)\left(1-\frac{h(\tau)}{r_0}\right). \tag{6}$$

The discretization of (5) by a central difference scheme gives rise to

$$\frac{V(\tau_i,\delta y) - V(\tau_i,-\delta y)}{2\delta y} = 0 \Rightarrow V(\tau_i,\delta y) = V(\tau_i,-\delta y).$$

An implicit finite difference scheme to (6) yields

$$\begin{aligned} V(\tau_i,\delta y) =& \frac{h(\tau_i)\delta y^2}{\sigma^2} m\left(1-e^{-r_0\tau_i}\right)\left(1-\frac{h(\tau_i)}{r_0}\right) + V(\tau_i,0) \\ =& m\left(1-e^{-r_0\tau_i}\right)\left[\frac{h(\tau_i)\delta y^2}{\sigma^2}\left(1-\frac{h(\tau_i)}{r_0}\right) + \frac{1}{r_0}\right]. \end{aligned} \tag{7}$$

To simplify notation, we let

$$a = \frac{\sigma^2 \delta\tau}{2\delta y^2 h(\tau)e^y} + \left[\frac{(\alpha - \beta h(\tau)e^y)}{h(\tau)e^y} + \frac{\sigma^2}{2h(\tau)e^y} + \frac{h'(\tau)}{h(\tau)}\right]\frac{\delta\tau}{2\delta y},$$
$$b = \left[\frac{1}{\delta\tau} - \frac{\sigma^2}{\delta y^2 h(\tau)e^y} + h(\tau)e^y\right]\delta\tau,$$
$$c = \frac{\sigma^2 \delta\tau}{2\delta y^2 h(\tau)e^y} - \left[\frac{(\alpha - \beta h(\tau)e^y)}{h(\tau)e^y} + \frac{\sigma^2}{2h(\tau)e^y} + \frac{h'(\tau)}{h(\tau)}\right]\frac{\delta\tau}{2\delta y}.$$

Applying implicit scheme to the PDE in (4) gives

$$V(\tau_i, y_j) = a(j)V(\tau_{i+1}, y_{j-1}) + b(j)V(\tau_{i+1}, y_j) + c(j)V(\tau_{i+1}, y_{j+1}) - m\delta\tau, \tag{8}$$

with

$$\frac{h'(\tau)}{h(\tau)} = \frac{h(\tau_i) - h(\tau_{i+1})}{\delta\tau h(\tau_i)}. \tag{9}$$

Then the problem becomes to solve for V such that $V(i,:)^T = A * V(i+1,:)^T + C$, where

$$A = \begin{pmatrix} e^{r_0\delta\tau} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a(1) & b(1) & c(1) & 0 & \cdots & 0 & 0 & 0 \\ 0 & a(2) & b(2) & c(2) & \cdots & 0 & 0 & 0 \\ 0 & 0 & a(3) & b(3) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b(j-2) & c(j-2) & 0 \\ 0 & 0 & 0 & 0 & \cdots & a(j-1) & b(j-1) & c(j-1) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}_{(J+1)*(J+1)}$$

and

$$C^T = \begin{pmatrix} \frac{m}{r_0}(1 - e^{r_0\delta\tau}) & -\delta\tau m & -\delta\tau m & -\delta\tau m & \cdots & -\delta\tau m & -\delta\tau m & 0 \end{pmatrix}_{1*(J+1)}.$$

Given $h(0) = h(\tau_0)$, we implement the iteration algorithm according to the following steps:

(1) Define matrix A and vector C.

(2) Start with an initial guess of $h(1)$, find the solution vector V by solving the above linear system. Calibrate the solution vector by shrinking $\delta\tau$ and δr and, or equivalently increasing I and J, until an prescribed error tolerance is reached (10^{-6}), thus get the

solution V matching the initial guess of $h(1)$. Compare the computed $V(\tau_1, \delta y)$ with the solution of $V(\tau_1, \delta y)$ provided in (7) (derived from (4)). Record the error between these two solutions.

(3) Calibrate $h(1)$ and repeat the solution procedure in step (2) until the recorded error is less than the prescribed tolerance level (10^{-4}).

(4) Repeat the above procedure to obtain $h(i)$ for $i = 2, 3, ..., n$.

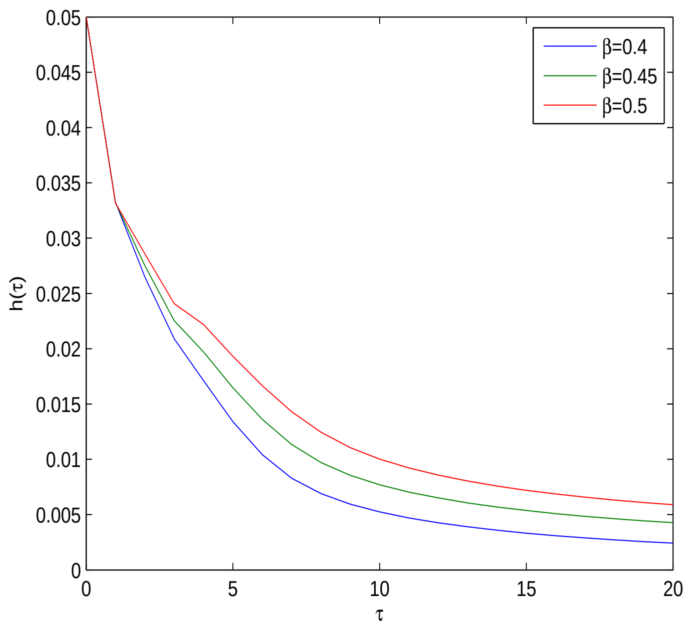

FIGURE 1. The optimal boundary of prepayment by variation of β.

Figure 1 and 2 display the numerical values of the optimal refinancing boundaries by variation of β and $\frac{\alpha}{\beta}$ accordingly. One may refer to [6, 17] for the choice of reasonable parameter values for the purpose of empirical test. One can see that $h(\tau)$ is the upper boundary for optimal prepayment, implying the debtor will prepay if the market rate r is less than $h(\tau)$. When CIR model is used for a market, it is theoretically assumed that the interest rate is always nonnegative. The market rate may never reach a theoretically computed negative $h(\tau)$. In this scenario, an optimal prepayment is not possible from the borrower's point of view. Figure 1 is an example plot of the optimal prepayment boundary for the mortgage contract.

3. A special case

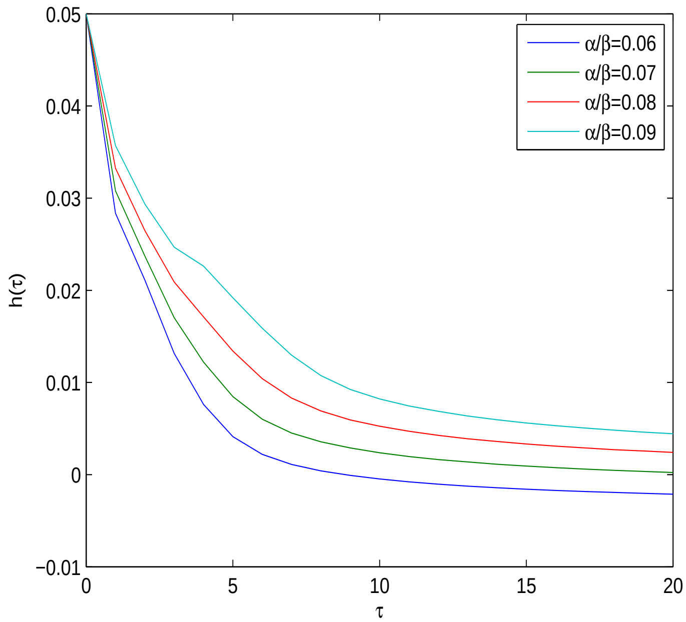

FIGURE 2. The optimal boundary of prepayment by variation of $\frac{\alpha}{\beta}$.

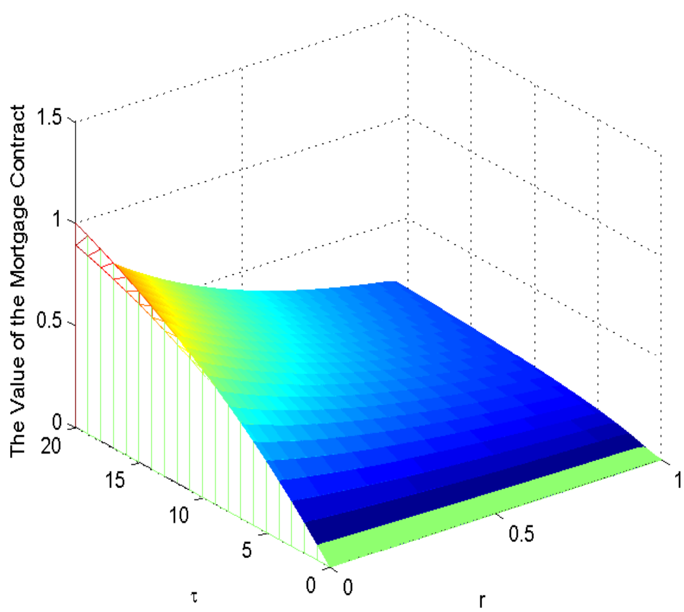

FIGURE 3. The value of $V(\tau, r)$ by implicit method with floating boundary, with $\beta = 0.5$, $\frac{\alpha}{\beta} = 0.07$ and $\sigma = 0.1$.

When $\sigma = 0$, the PDE in (2) is reduced to the following one:

$$\frac{\partial V}{\partial \tau} - (\alpha - \beta r)\frac{\partial V}{\partial r} + rV = m. \tag{10}$$

The explicit solution using the characteristic function approach for small σ is given by (see [19])

$$V(r,\tau) = me^{-\frac{r-\frac{\alpha}{\beta}}{\beta}} \int_0^\tau e^{-\frac{\alpha}{\beta}s+\frac{r-\frac{\alpha}{\beta}}{\beta}} ds.$$

On the boundary $h(\tau)$, the value of the contract is known as $V = \frac{m}{r_0}(1-e^{-r_0\tau})$. Define

$$Q = V(r,\tau) - V(h(\tau),\tau) = me^{-\frac{r-\frac{\alpha}{\beta}}{\beta}} \int_0^\tau e^{-\frac{\alpha}{\beta}s+\frac{r-\frac{\alpha}{\beta}}{\beta}} ds - \frac{m}{r_0}\left(1-e^{-r_0\tau}\right),$$

$h(\tau)$ can be obtained by solving $Q = 0$ numerically.

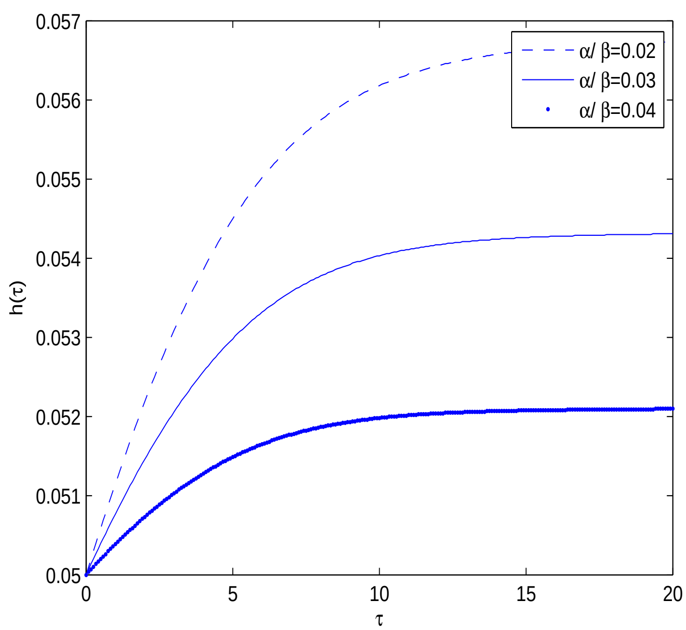

FIGURE 4. The optimal boundary when $r_0 > \frac{\alpha}{\beta}$.

We restrict our attention to a rectangular mesh around (t_j, r_m) with uniform time step size $\delta t = t_{j+1} - t_j$ and space mesh size $\delta r = r_{j+1} - r_j$ for the case when $\sigma \to 0$. Let $c = -(\alpha - \beta r_m) = -(\alpha - \beta(r_0 + m\delta r))$. Recall that

$$\frac{\partial V}{\partial r}|_{(\tau_j, r_m)} = \frac{V(\tau_j, r_{m+1}) - V(\tau_j, r_m)}{\delta r},$$

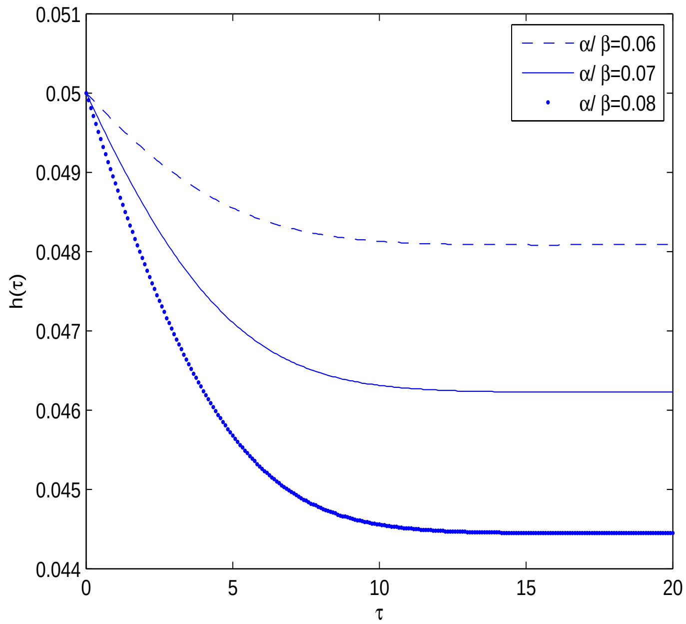

FIGURE 5. The optimal boundary when $r_0 < \frac{\alpha}{\beta}$.

$$\frac{\partial V}{\partial \tau}|_(\tau_j, r_m) = \frac{V(\tau_{j+1}, r_m) - V(\tau_j, r_m)}{\delta \tau}.$$

Substitute the above two expressions into (10), we have

$$V(\tau_{j+1}, r_m) = [1 + \frac{c\delta t}{\delta r} - \delta\tau(r_0 + m\delta r)]V(\tau_j, r_m) - \frac{\delta\tau c}{\delta r}V(\tau_j, r_{m+1}).$$

To simplify our notation, we let $\eta = \frac{c\delta t}{\delta r}$, and thus

$$V(\tau_{j+1}, r_m) = [1 + \eta - \delta\tau(r_0 + m\delta r)]V(\tau_j, r_m) - \eta V(\tau_j, r_{m+1}). \tag{11}$$

To meet the stability requirement, let

$$V(\tau_i, r_m) = e^{ikr_m}$$

and

$$V(\tau_{i+1}, r_m) = \lambda e^{ikr_m},$$

where λ is the magnification factor. Substituting into (11) leads to

$$\lambda e^{ikr_m} = [1 + \eta - \delta\tau(r_0 + m\delta r)]e^{ikr_m} - \eta\lambda e^{ikr_m} e^{ik\delta x}$$

The von Neumann stability (see [11]) requires that $|\lambda|^2 \leq 1$, or equivalently,

$$(1+\eta-\delta\tau(r_0+m\delta r))^2-2(1+\eta-\delta\tau(r_0+m\delta r))\eta\cos(k\delta r)+\eta^2 \leq 1,$$

where further algebraic implication may be possible, but not necessary for checking specific algorithms. To prove the consistency of the implicit scheme, note that

$$\frac{V(\tau_{j+1},r_m)-V(\tau_j,r_m)}{\delta\tau}-(\alpha-\beta r)\frac{V(\tau_j,r_{m+1})-V(\tau_j,r_m)}{2\delta r}+rV(\tau_{j+1},r_m)=0.$$

Let

$$p_{\delta\tau,\delta r}\phi=\frac{\phi(\tau_{j+1},r_m)-\phi(\tau_j,r_m)}{\delta\tau}-(\alpha-\beta r)\frac{\phi(\tau_j,r_{m+1})-\phi(\tau_j,r_m)}{2\delta r}+r\phi(\tau_{j+1},r_m).$$

As

$$\phi(\tau_j,r_{m\pm1})=\phi(\tau_j,r_m)\pm\delta r\phi_r+\frac{1}{2}\delta r^2\phi_{rr}\pm\frac{1}{6}\delta r^3\phi_{rrr}+o(\delta r),$$

thus, we have

$$\frac{\phi(\tau_j,r_{m+1})-\phi(\tau_j,r_m)}{2\delta r}=\delta r\phi_r+\frac{1}{6}\delta r^3\phi_{rrr}+o(\delta r),$$

which implies

$$p_{\delta\tau,\delta r}\phi=\phi_\tau-(\alpha-\beta r)\phi_r+r\phi(\tau_j,r_{m+1})-(\alpha-\beta r)\frac{1}{6}\delta r^3\phi_{rrr}-\frac{\phi(\tau_j,r_m)}{\delta\tau}-\frac{\delta r^2\phi_{rr}}{2\tau}.$$

The method is consistent as $\delta r,\delta\tau\to0$. To numerically demonstrate the convergence of the method, we first notice that the results of V tend to be stationary as $r\to r_{max}$. Recall that one of the boundary conditions is $V(\tau,r_{max})=0$, which means the $V\to0$ when $r_{max}\to\infty$. But the upper boundary is set as $r_{max}=1$ in our computation, which may affect the accuracy of the results. To make comparisons with existing explicit solutions, we focus on the case when $\sigma=0$. The following Table 1 records the errors between the exact solution and the solution using the finite difference scheme.

4. Concluding remark

This paper focuses on the numerical approach for valuing the mortgage contract and obtaining the best strategy for debtors to make prepayment. Finite difference method is proposed and designed to solve the governing partial differential equations. The finite difference schemes are

$V(r_{J-1},\tau)$	$J=64$	$J=128$	$J=256$	$J=512$
$V(r_{J-1},1)$	8.029883641	7.969268381	7.943120935	7.937716918
$V(r_{J-1},2)$	8.038035387	7.977297518	7.951097461	7.945682554
$V(r_{J-1},3)$	8.038043663	7.977305607	7.951105471	7.945690548
$V(r_{J-1},4)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},5)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},6)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},7)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},8)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},9)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},10)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},11)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},12)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},13)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},14)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},15)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},16)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},17)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},18)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},19)$	8.038043671	7.977305615	7.951105479	7.945690556
$V(r_{J-1},20)$	8.038043671	7.977305615	7.951105479	7.945690556

TABLE 1. Errors for different grid J by implicit method (in 10^{-5}) , when $\delta\tau = 1$.

tested with both fixed boundaries and floating boundaries for the problem. Numerical experiments are provided and compared with analytical results for small market volatility environment. The method is robust and can be applied to other option valuation problems. As one of the possible future directions, it would be interesting to incorporate the housing value into the model and investigate the collective and interactive effects of interest rate and property value on mortgage price.

Conflict of Interests

The authors declare that there is no conflict of interests.

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