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EXTENSION OF THE MILLER AND MODIGLIANI THEORY TO ALLOW FOR SHARE REPURCHASES

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Abstract: Miller and Modigliani (1961) consider valuation of infinite horizon firms that may not engage in purchasing their own shares. While their fundamental valuation approach also applies to firms that purchase their own shares, their stream of dividends approach does not apply to these firms if they do not distribute "sufficient" cash via dividends and share repurchases, as characterized by a necessary and sufficient condition. Also presented is a modified stream of dividends approach that provides an equivalent valuation of every firm that can be valued by the fundamental approach.

Keywords: valuation; share price; dividend approach; cash flow approach; MM theory; share repurchase.

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In Section I of their seminal paper, Miller and Modigliani (1961), designated as MM hereafter, provide a valuation formula for an infinite horizon firm under the assumption of an ideal world characterized by perfect capital markets, rational behavior, and perfect certainty. The valuation is based on the fundamental principle that the price of each share must be such that the rate of return (dividend plus capital gains per dollar invested) on every share will be the same throughout the market over any given interval of time.

In Section II of their paper, MM claim that a popular valuation approach called the stream of dividends approach is equivalent to the fundamental valuation approach used in Section I. It is important to point out that their proof of equivalence on pages 422-423 of their paper is incomplete in the sense that it proves only that the dividend stream approach implies the fundamental approach and it does not prove the converse, namely, that the fundamental

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approach implies the stream of dividends approach. Furthermore, we should point out that while it is straightforward – and perhaps the reason for MM's omission of it – to complete the proof when the firm may only issue new equity, as carried out in Sethi et al. (1982, 1991) are significantly more complicated when the firm may also repurchase its own shares. More specifically, their dividend stream formula (13) needs to be subtly modified if the equivalence is to be restored.

We should mention further that in Footnote 12 of their paper, MM do mention that their equivalence result does not extend to certain pathological extreme cases, including the obvious example of a legendary company that will never pay any dividend, fortunately of no real economic significance. However, this legendary firm can acquire economic significance, if it is able to distribute "sufficient" cash to stockholders by way of share repurchases; Example 2 in Section 2 is a case in point. Generally speaking, MM's equivalence continues to hold for firms that are able to distribute "sufficient" cash by way of dividends or share repurchases or both.

Yet there remain firms, like the one in Example 3 in Section 2, that are valued by the fundamental approach, but cannot be valued by the dividend approach. Roughly speaking, these are firms that do not distribute "sufficient' cash to stockholders via the two mechanisms. Whether or not to call such firms pathological is a matter of preference, since the fundamental approach can still value them. Were these to be converted to proprietorships so that only the cash flows mattered, we would not consider them pathological. What really happens with these firms is that the corporate mechanism is not able to distribute "sufficient" cash via dividends and share repurchases, and thus leaves a positive liquidation value per share at "infinity." The logic here is very subtle, since the firm by assumption must have a zero value at infinity. A little reflection tells us that there must be "zero shares at infinity," so that while the firm reaches a zero valuation asymptotically, the limiting share price is of the 0/0 indeterminate form having a positive limit. But having "zero shares at infinity" is only a necessary condition for such firms, as is clear by the firm in Example 4 which distributes "sufficient" cash. The important question therefore is how to precisely characterize firms that distribute "sufficient" cash. Sethi et al. (1982, 1991) and Sethi (1996) answer this question by providing a *necessary and sufficient condition* (2.13) that characterizes such firms.

Another question is whether it is possible to modify the dividend stream approach of valuation of infinite horizon firms including those which may engage in purchasing their own shares, such that the modified approach is equivalent to the fundamental valuation approach for all firms including those that do not distribute "sufficient" cash. The answer is yes and it comes from the way we value finite horizon firms with horizon $T > 0,$ that is finite or almost surely finite, if random. In valuing these firms, the difficulty that one faces is to come up with a specification of the value of the firm at time T. Since any such specification is likely to be arbitrary in many cases, which of course would affect a firm's valuation as well as its share price at all earlier times, there may be some preference to treat infinite horizon firms. In any case, Miller and Modigliani consider infinite horizon firms in their 1961 paper, and our purpose is to extend their analysis to allow for share repurchases by such firms.

Given that it is a necessary and sufficient condition it must accompany any extension of the MM theory to allow for share repurchases. Notwithstanding, this condition is still not well known and, to my knowledge, not mentioned in any existing finance texts that devote at least a chapter covering the MM theory. The purpose of this paper is to exposit the issues that relate to extending the MM's equivalence proposition to firms that may repurchase its own shares, the important role of the necessary and sufficient condition characterizing "sufficient" cash distribution, and modifying the traditional stream of dividends approach to restore the equivalence proposition to all firms. We also include a new discussion of how we can bring in a stock split procedure in order to have a "positive number of outstanding shares at infinity" instead of zero shares in the limit. All this is done in the framework of MM including their notations to facilitate a comparison to MM.

The plan of the paper is as follows. In the next section, we describe the fundamental valuation approach of MM, which applies to all firms under consideration. Section 2 shows, along with four illustrative examples, that the traditional stream of dividends approach as defined in MM does not extend to a class of firms that may purchase their own shares. In Section 3 we provide the modified dividend stream approach and prove its equivalence to the fundamental valuation approach. Section 4 concludes the paper.

1. The Fundamental Valuation Approach

We begin by recalling the notation used by MM and by introducing some additional notation.

Let

$d(t) =$ dividends per share paid by the firm during period t, $t = 0,1,\ldots,$

$p(t) =$ the price (ex any dividend in t -1) of a share in the firm at the start of period t,

$n(t) =$ the number of shares of record at the start of t, with $n(0)$ given,

$m(t+1) =$ the number of shares (if any) sold during t at the ex-dividend closing price $p(t+1)$ so that

$n(t+1) = n(t) + m(t+1),$

$V(t) = n(t)p(t) =$ the total value of the firm at the start of period t,

$D(t) = n(t)d(t) =$ the total dividends paid in period t to the holders of record at the start of t,

$E(t) = m(t+1)p(t+1) =$ the total equity capital raised during period t,

$X(t) =$ the firm's net earnings during period t,

$I(t) =$ the firm's investment during period t,

$\rho =$ the required rate of return in each period; it is assumed to be constant for simplicity in exposition and it will also be referred to as the discount rate.

Using the fundamental principle of valuation

$$p(t) = \frac{1}{1+\rho}[d(t) + p(t+1)], \tag{1.1}$$

some bookkeeping identities, and the transversality condition

$$\frac{1}{(1+\rho)^T} V(T) \to 0 \text{ as } T \to \infty, \tag{1.2}$$

MM obtain the value of the firm $V(0)$ at time 0 to be

$$V(0) = \sum_{\tau=0}^{\infty} \frac{1}{(1+\rho)^{\tau+1}}[X(\tau) - I(\tau)]. \tag{1.3}$$

Here we see that for discounting purposes, $d(t)$ is assumed to be paid at the end of period t, so that it gets the same discounting treatment as $p(t+1)$. Likewise, $D(t)$, $E(t)$, $m(t)$, $X(t)$ and $I(t)$ are assumed to occur at the end of period t, and $V(t)$ and $n(t)$ are evaluated at the beginning of period t.

We should mention that transversality conditions like (1.2) are quite common in the economics literature in ruling out the so-called *bubbles*, and we will not consider the issue relating to bubbles any further in this paper.

Likewise, as in (1.3), it is easy to obtain the value of the firm at time t as

$$V(t) = \sum_{\tau=0}^{\infty} \frac{1}{(1+\rho)^{\tau+1}} [X(t+\tau) - I(t+\tau)] .$$

MM also mention that if *I(t)* is the given level of the firm's investment or increase in its holding of physical assets in period *t* and if *X(t)* is the firm's total net profit for the period, then the amount of outside capital *E(t)* required will be equal to *I(t)* - [*X (t)* -*D (t)*]. We can therefore rewrite the above valuation formula as

$$V(t) = \sum_{\tau=0}^{\infty} \frac{1}{(1+\rho)^{\tau+1}} [D(t+\tau) - E(t+\tau)]. \tag{1.4}$$

Furthermore, the number of outstanding shares and the share price, respectively, can be easily derived as

$$n(t) = n(0) \prod_{\tau=0}^{t-1} \left[1 - \frac{E(\tau)}{V(\tau+1)} \right]^{-1} \text{ and } p(t) = \frac{V(t)}{n(t)}. \tag{1.5}$$

Before proceeding to the next section, let us note that the derivation of formulas (1.3) and (1.4) do not depend on the sign of $m(t+1)$ or of $E(t)$. Thus, if we were to interpret a negative value of $m(t+1)$ as repurchase of $-m(t+1)$ shares by the firm for a total price of $-E(t) = -m(t+1)p(t+1)$, all we have to do in Section I of the MM paper is to assume the number of shares repurchased to be less than the shares outstanding. That is,

$$-m(t+1) \le n(t), \tag{1.6}$$

so that $n(t+1) = n(t) + m(t+1)$ remains positive. Note that $-m(t+1) = n(t)$ would imply a liquidation of the firm with the consequence that $V(t+1) = 0$.

Since the derivation of the formula of the firm’s value does not depend on the sign of $m(t+1)$ or of $E(t)$, the fundamental valuation approach continues to hold. So henceforth, we consider all firms subject to the conditions (1.2) and (1.6).

Looking at (1.4) and (1.5), we see that all we need to derive them are $D(t)$, $E(t)$, and ρ. So, we can define a firm by the triple $\{D(t), E(t), \rho\}$ for each t, and we assume all of this information is known with certainty right from the beginning, and that it is the only information we have. We thus do not consider the issue of the market inferring some additional information associated with the dividends and share repurchases over time. In the case when dividends and share repurchases carry additional information, one needs to make a stochastic extension of the MM theory, and we refer the reader to Sethi et al. (1991, 1996) for such extensions.

2. The Stream of Dividends Approach and Share Repurchase

While the fundamental valuation approach in Section I of the MM paper holds regardless of whether $m(t+1)$ is positive or negative, the dividend stream approach runs into trouble when $m(t+1)$ may take negative values. Before proceeding further, we recall that MM define the dividend stream approach by their equation (13), which is

$$p(t) = \sum_{\tau=0}^{\infty} \frac{d(t+\tau)}{(1+\rho)^{\tau+1}} \tag{2.1}$$

along with the bookkeeping identities (1.5) and the transversality condition (1.2). MM show that the dividend stream approach, in the absence of share repurchase, implies the fundamental valuation approach, however, they do not show the reverse implication.

Let us try to complete the omitted part of MM's equivalence proof by starting with the fundamental approach (1.1), and show how we can end up with (2.1). By recursion of (1.1), it is easy to obtain

$$p(t) = \sum_{\tau=0}^{T-1-t} \frac{d(t+\tau)}{(1+\rho)^{\tau+1}} + \frac{1}{(1+\rho)^{T-t}}\, p(T). \tag{2.2}$$

Using (1.4) and the relation $V(T) = p(T)n(T)$ in (2.2), we obtain

$$p(t) = \sum_{\tau=0}^{T-1-t} \frac{d(t+\tau)}{(1+\rho)^{\tau+1}} + \frac{1}{(1+\rho)^{T-t}} \frac{V(T)}{n(T)}. \tag{2.3}$$

Taking the limit as $T \to \infty$, we obtain

$$p(t) = \sum_{\tau=0}^{\infty} \frac{d(t+\tau)}{(1+\rho)^{\tau+1}} + \frac{1}{(1+\rho)^{-t}} \lim_{T\to\infty} \frac{1}{(1+\rho)^{T}} \frac{V(T)}{n(T)}. \tag{2.4}$$

In particular,

$$p(0) = \sum_{\tau=0}^{\infty} \frac{d(\tau)}{(1+\rho)^{\tau+1}} + \lim_{T\to\infty} \frac{1}{(1+\rho)^{T}} \frac{V(T)}{n(T)}. \tag{2.5}$$

Clearly, if $m(t+1) \geq 0$ for all t, the limit in the second term of the RHS of (2.5) would vanish on account of (1.2) and $n(T) \geq n(0) > 0$, and MM's equivalence proof would be completed in the case of no share repurchase.

But if the firm engages in share repurchases, $n(T) \geq n(0) > 0$ can no longer be guaranteed and the equivalence would hold if and only if the limit term on the RHS of (2.5) would vanish. At this point, it would be worthwhile to provide examples of four firms illustrating why the limit might not vanish in some cases of firms with repurchases allowed. All four firms have the same cash flow streams, so they have the same value. Yet, the way in which they allocate these cash flows to dividends and repurchases determines whether or not the share price represents the present value of the dividends accruing to a share.

Example 1. Firm 1 pays dividends, issues no new stock, and does not repurchase its own shares. More specifically, let

$$n_1(0) = 100,\ D_1(t) = \left(\frac{1+\rho}{2}\right)^{t+1},\ E_1(t) = 0.$$

Using formula (1.4) of the fundamental valuation approach gives

$$V_1(t) = \sum_{\tau=0}^{\infty} \frac{1}{(1+\rho)^{\tau+1}} \left[\left(\frac{1+\rho}{2}\right)^{t+\tau+1}\right] = \left(\frac{1+\rho}{2}\right)^{t}. \tag{2.6}$$

Also $n_1(t) = n_1(0) = 100.$ Then, $V_1(0) = 1$ and $p_1(0) = \dfrac{V_1(0)}{n_1(0)} = \dfrac{1}{100}$, and since

$d_1(t) = \dfrac{D_1(t)}{n_1(t)} = \dfrac{1}{100}\left(\dfrac{1+\rho}{2}\right)^{t+1}$, and therefore $\displaystyle\sum_{\tau=0}^{\infty} \frac{d_1(\tau)}{(1+\rho)^{\tau+1}} = \frac{1}{100}\sum_{\tau=0}^{\infty}\left(\frac{1}{2}\right)^{\tau+1} = \frac{1}{100}$, the share price $p_1(0)$ is also the present value of the dividends accruing to each share.

Example 2. Firm 2 never pays any dividend but repurchases its own shares. More specifically, let

$$n_2(0) = 100,\ D_2(t) = 0,\ E_2(t) = -\left(\frac{1+\rho}{2}\right)^{t+1}.$$

Then using (1.4), we have $V_2(t) = \left(\dfrac{1+\rho}{2}\right)^{t}$ as in Example 1. Thus $p_2(0) = \dfrac{V_2(0)}{n_2(0)} = \dfrac{1}{100}$. But, the present value of the dividends per share in this example is zero, whereas $p_2(0) = \dfrac{1}{100} > 0.$

Let us now find $p_2(0)$ by using formula (2.5) of the modified dividend approach. First, from (1.5), we see that

$$n_2(t) = 100\prod_{\tau=0}^{t-1}\left[1+\frac{\left(\frac{1+\rho}{2}\right)^{t+1}}{\left(\frac{1+\rho}{2}\right)^{t+1}}\right]^{-1} = 100\prod_{\tau=0}^{t-1}\frac{1}{2} = 100\left(\frac{1}{2}\right)^t. \tag{2.7}$$

Since $d_2(t) = \dfrac{D_2(t)}{n_2(t)} = 0,$ we have from (2.5),

$$p_2(0) = 0 + \lim_{T\to\infty}\frac{1}{(1+\rho)^T}\frac{V_2(T)}{n_2(T)} = \lim_{T\to\infty}\frac{1}{(1+\rho)^T}\frac{\left(\frac{1+\rho}{2}\right)^T}{100\left(\frac{1}{2}\right)^T} = \frac{1}{100}, \tag{2.8}$$

which is the same as the price obtained by the fundamental valuation approach.

Example 3. Firm 3 pays a positive dividend and repurchases shares. Specifically,

$$n_3(0) = 100,\ D_3(t) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^t(1+\rho)^{t+1} > 0,\ E_3(t) = \frac{1}{2}\left[\left(\frac{1}{4}\right)^t - \left(\frac{1}{2}\right)^t\right](1+\rho)^{t+1} < 0.$$

Since $D_3(t) - E_3(t) = \left(\dfrac{1+\rho}{2}\right)^{t+1}$, which is the same cash flow as in the previous two examples, the value of the firm $V_3(t) = \left(\dfrac{1+\rho}{2}\right)^t$. Likewise, $p_3(0) = \dfrac{V_3(0)}{n_3(0)} = \dfrac{1}{100}$.

Next, we use (1.5) to obtain

$$n_3(t) = 100\prod_{\tau=0}^{t-1}\left[1-\frac{\frac{1}{2}\left[\left(\frac{1}{4}\right)^\tau - \left(\frac{1}{2}\right)^\tau\right](1+\rho)^{\tau+1}}{\left(\frac{1+\rho}{2}\right)^{\tau+1}}\right]^{-1}$$

$$= 100\prod_{\tau=0}^{t-1}\left[1-2^\tau\left[\left(\frac{1}{4}\right)^\tau - \left(\frac{1}{2}\right)^\tau\right]\right]^{-1} = \prod_{\tau=0}^{t-1}\frac{100}{2-\left(\frac{1}{2}\right)^\tau}. \tag{2.9}$$

Then,

$$d_3(t) = \frac{D_3(t)}{n_3(t)} = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^t(1+\rho)^{t+1}\frac{1}{100}\prod_{\tau=0}^{t-1}\left(2-\frac{1}{2^\tau}\right). \tag{2.10}$$

We can now use $V_3(t)$ and $n_3(t)$ to compute the second term on the RHS of (2.5):

$$\lim_{T\to\infty}\frac{1}{(1+\rho)^T}\frac{V_3(T)}{n_3(T)}=\lim_{T\to\infty}\left(\frac{1}{2}\right)^T\frac{1}{100}\prod_{\tau=0}^{T-1}\left(2-\frac{1}{2^\tau}\right)=\lim_{T\to\infty}\frac{1}{100}\prod_{\tau=0}^{T-1}\frac{1}{2}\left(2-\frac{1}{2^\tau}\right)$$
$$=\lim_{T\to\infty}\frac{1}{100}\prod_{\tau=0}^{T-1}\left(1-\frac{1}{2^{\tau+1}}\right)=\frac{1}{100}\prod_{k=0}^{\infty}\left(1-\frac{1}{2^k}\right)=\frac{Q}{100}, \tag{2.11}$$

where $Q=0.2887880950...$ is a positive constant encountered in digital tree searching (Flajolet and Richmond, 1992); infinite product in (2.11) is known as the Euler function with a variable q in place of 1/2. Next, we compute the first term of the RHS of (2.5), the present value of dividends accruing to a share. Using (2.10),

$$\sum_{\tau=0}^{\infty}\frac{d_3(\tau)}{(1+\rho)^{\tau+1}}=\sum_{\tau=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^\tau\frac{1}{100}\prod_{k=0}^{\tau-1}\left(2-\frac{1}{2^k}\right)=\frac{1}{100}\sum_{\tau=0}^{\infty}\frac{1}{2^{\tau+1}}\prod_{k=0}^{\tau-1}\left(1-\frac{1}{2^{k+1}}\right)$$
$$=\frac{1}{100}\sum_{\tau=1}^{\infty}\frac{1}{2^\tau}\prod_{k=1}^{\tau-1}\left(1-\frac{1}{2^k}\right)=\frac{(1-Q)}{100}. \tag{2.12}$$

Since the initial price formula (2.5) is derived from the fundamental valuation approach (1.1), (1.2), and (1.5), we know that the sum of the two terms obtained in (2.11) and (2.12), respectively, equals the initial share price $p_3(0)=\dfrac{1}{100}$. Accordingly, we can claim that the expression derived in (2.12) equals $\dfrac{(1-Q)}{100}$. A more direct proof follows from an identity for q-Pochhammer symbols (http://mathworld.wolfram.com/q-PochhammerSymbol.html) established as Propositio II in Euler (1760) and translated by Bell (2009). Numerically, we have verified this by computing the two terms in (2.11) and (2.12) to 50 digit precision, and their sum at that precision level equals 1.

Example 4. Firm 4 distributes equal amounts via dividends and repurchases. Specifically,

$$n_4(0)=100,\ D_4(t)=\left(\frac{1}{2}\right)\left(\frac{1+\rho}{2}\right)^{t+1},\ E_4(t)=-\frac{1}{2}\left(\frac{1+\rho}{2}\right)^{t+1}.$$

Clearly $D_4(t)-E_4(t)=\left(\dfrac{1+\rho}{2}\right)^{t+1}$, which is the same cash flow as in the previous examples, and so the value of the firm $V_4(t)=\left(\dfrac{1+\rho}{2}\right)^{t}$. This gives $V_4(0)=1$ and $p_4(0)=\dfrac{V_4(0)}{n_4(0)}=\dfrac{1}{100}$.

From (1.5), $n_4(t) = 100\prod_{\tau=1}^{t-1}\left(\frac{3}{2}\right)^{-1} = 100\left(\frac{2}{3}\right)^t$. Then, the per share dividend $d_4(t) = \frac{D_4(t)}{n_4(t)} = \frac{1}{2}\frac{1}{100}\left(\frac{3}{2}\right)^t\left(\frac{1+\rho}{2}\right)^{t+1}$. We can now use these to see that the first term on the RHS of (2.5) is

$$\sum_{\tau=0}^{\infty}\frac{d_4(\tau)}{(1+\rho)^{\tau+1}} = \frac{1}{400}\sum_{\tau=0}^{\infty}\left(\frac{3}{4}\right)^{\tau} = \frac{1}{100},$$

and the second term on the RHS is

$$\lim_{T\to\infty}\frac{1}{(1+\rho)^T}\frac{V_4(T)}{n_4(T)} = \lim_{T\to\infty}\frac{1}{100}\left(\frac{3}{4}\right)^T = 0.$$

Let us summarize our observations. Firm 1 pays only dividends, and its number of outstanding shares $n_1(t) = 100 > 0$. For this firm, the share price is the present value of the dividends accruing to a share. Firm 2 pays no dividends and repurchases shares. Firms 3 and 4 pay dividends as well as repurchase shares. But the limit in the second term of the RHS of (2.5) does not vanish for Firms 2 and 3, and does vanish for Firm 4. For these three firms, it can be easily shown that $n_2(t) \to 0,$ $n_3(t) \to 0,$ and $n_4(t) \to 0$ as $t \to \infty$.

The observation that there are "no outstanding shares in the limit" might lead one to believe that these firms are not realistic in some sense. The fact of the matter is that these firms are just as realistic as any other infinite horizon firm. To see this, let us first note that $n_2(t) > 0,$ $n_3(t) > 0,$ and $n_4(t) > 0,$ for every t. Second, it is easy to introduce stock splits so that the number of outstanding shares do not approach zero as $t \to \infty$. For instance, a two-to-one stock split at the beginning of each period $t = 0, 1, 2, \ldots,$ in Example 2 will make the number of after-split shares to be 200. During each period, the dividends will be paid and then half of the shares will be repurchased at the ex-dividend price. In this case, the outstanding number of shares at the beginning of each period before the split will be 100 and after the split will be 200. With this, Firm 2 in Example 2 can be changed to an equivalent firm whose outstanding number of shares do not approach zero in the limit. Of course, if we allow splits, we have to change the $n(t)$ and $p(t)$ formulas in an appropriate manner, and that is easy to do.

What is of essence in Examples 2, 3 and 4 is that the residual share price "at infinity" vanishes for Firm 4 and not for Firms 2 and 3. As we shall see below, Firm 4 gives dividends that do not fall short in relation to its share price valuation, whereas Firm 2 gives no dividends and Firm 3 gives dividends that fall short in relation to their respective share price valuations, and as a result there is a positive residual share price "at infinity" for Firms 2 and 3. In other words, these firms do not distribute "sufficient" cash by way of dividends and share repurchases, and leave some positive cash per share on the table "at infinity." If we were given a residual liquidation value in the case of a finite horizon firm, no one would question the presence of the terminal term in the share price equation (2.3). What most people find hard to believe is that this terminal term can also occur in the case of an infinite horizon firm depending on its dividend and equity repurchase policies.

The natural question that arises is how to characterize a firm's policies $D(t)$ and $E(t)$, $t = 0,1,...,$ for which the limit term on the RHS of (2.5) vanishes. Sethi, Derzko and Lehoczky (1982, 1991) have studied this question. In the present context, they show that

$$\lim_{T\to\infty} \frac{1}{(1+\rho)^T} \frac{V(T)}{n(T)} = 0 \text{ iff } \sum_{\tau=0}^{\infty} \frac{D(\tau)}{V(\tau)} = \infty, \tag{2.13}$$

where we recall that $V(t),\ t = 1,2,...,$ is known by formula (1.4) given $D(t)$ and $E(t)$, $t = 0,1,...$. Note that the dividend yield in each period is simply a ratio of two quantities with dollar units. It therefore does not require any discounting when we sum such ratios over the infinite horizon.

Because of this complete characterization via a necessary and sufficient condition, there is no escaping the conclusion that the second term on the RHS of (2.5) will not vanish iff $\sum_{\tau=0}^{\infty} \frac{D(\tau)}{V(\tau)} < \infty,$ i.e., the sum of the dividend yields is finite. Thus, a firm may be considered to be distributing "sufficient" cash to its stockholders in relation to its valuation iff $\sum_{\tau=0}^{\infty} \frac{D(\tau)}{V(\tau)} = \infty$, and to be distributing "insufficient" cash in relation to its valuation iff $\sum_{\tau=0}^{\infty} \frac{D(\tau)}{V(\tau)} < \infty.$ It is easy to show that Firms 1 and 4 in our examples are distributing "sufficient" cash, whereas Firms 2 and 3 are not. These results raise a challenging issue of whether the share price of a firm, like Firms 2 and 3, will be undervalued by the investors who price shares based only on their dividend contents. This is not a question based on rationality, since the fundamental approach correctly provides the share prices even for such firms. It is more a question of whether these investors are behaving irrationally in valuing the shares of such firms by their divided contents.

Another equivalent characterization of this condition is derived in Sethi (1996). It is shown that cash distributions are “insufficient” iff a share in the dividend reinvestment plan grows to only a finite number of shares “at infinity.”

Now that we have demonstrated that the traditional dividend stream approach cannot value firms in some cases when share repurchase is allowed, our next task is to value all possible firms including Firms 2 and 3 that distribute “insufficient” cash. Clearly and specifically, the stream of dividends approach applied to such firms *cannot* yield the share price formula (2.1). It is imperative, therefore, to modify the traditional stream of dividends approach so that it can be made equivalent to the fundamental valuation approach. This is carried out in the next section.

3. The Modified Dividend Streams Approach and Proof of Equivalence

We begin with a discussion of Firm 2 in Example 2. The firm pays no dividends, distributes cash only by share repurchases, and has a strictly positive share price determined by the fundamental approach. Suppose the firm decides to liquidate at some time T, then it would pay a liquidating dividend equal to the value of its share at that time, effectively “purchasing” all of its shares at that time. Keep in mind that the share repurchases, as obtained in Example 2, do not amount to liquidation at any finite time except in the limit “at infinity.”

So, in order to develop a modified dividend stream approach that can correctly price a firm’s shares, let us do the following thought experiment. Consider a firm, defined by the triple $\{D(t), E(t), \rho\}$ for each t, that decides to liquidate at some distant finite time T by giving a liquidating dividend equaling the share price at that time. From the fundamental approach, we then know that the share price $p(t)$ can be expressed by formula (2.3), where the second term on the RHS is nothing but the liquidating dividend. As T increases, the formula continues to hold good. In other words, $p(t)$ defined in (2.3) remains the same no matter when the liquidating dividend is paid out, as long as the amount of the dividend in period T is $\frac{V(T)}{n(T)}$ determined for each T in Section 1 by using the fundamental approach. Clearly

$p(t)$ will remain the same if we take the limit of the liquidating dividend in (2.3) as $T \to \infty$ and obtain (2.4), which we rewrite as

$$p(t) = \sum_{\tau=0}^{\infty} \frac{d(t+\tau)}{(1+\rho)^{\tau+1}} + (1+\rho)^t \lambda, \tag{3.1}$$

where the limiting per share liquidating dividend in the present-value terms is

$$\lambda = \lim_{T\to\infty} \frac{1}{(1+\rho)^T} \frac{V(T)}{n(T)}. \tag{3.2}$$

Now if we limit ourselves only to the fundamental valuation approach, we obtain the value $V(t)$ of the infinite horizon firm in formula (1.4) and the share price in (1.5) or (3.1), and there is nothing more to be done.

But if we are to develop the modified dividend stream approach and show it to be equivalent to the fundamental approach, there is no alternative but to take a clue from (2.3) and define

$$p(t) = \lim_{T\to\infty} \left[\sum_{\tau=0}^{T-1-t} \frac{d(t+\tau)}{(1+\rho)^{\tau+1}} + \frac{1}{(1+\rho)^{T-t}} \frac{V(T)}{n(T)} \right], \tag{3.3}$$

where $V(T)$ at any given time T is assumed to be *not known* and is to be determined by using (3.3), (1.2), and (1.5), which we shall refer to as the modified dividend stream approach in contrast to the traditional dividend stream approach defined by (2.1), (1.2) and (1.5).

From the earlier discussion, one could interpret (3.3) to be the share price of a firm that would "eventually" distribute an appropriate liquidating dividend to be determined.

We can now prove the following result.

Theorem 3.1. *The modified dividend stream approach defined by* (3.3), (1.2) *and* (1.5) *is equivalent to the fundamental valuation approach defined by* (1.1), (1.2) *and* (1.5).

Proof. Since (2.4) was obtained by the fundamental approach, it is immediate that the fundamental approach implies (3.3) but not (2.1) in general and, therefore, the modified dividend stream approach.

To go the other way, it is easy to see from (3.3) that

$$p(t) - \frac{1}{1+\rho} p(t+1) = \lim_{T\to\infty} \left[\sum_{\tau=0}^{T-1-t} \frac{d(t+\tau)}{(1+\rho)^{\tau+1}} - \sum_{\tau=0}^{T-2-t} \frac{d(t+1+\tau)}{(1+\rho)^{\tau+2}} \right] = \frac{d(t)}{1+\rho}, \tag{3.4}$$

which is nothing but (1.1). This completes the proof since relations (1.2) and (1.5) are common to both approaches.

That (3.3) is the correct replacement as well as generalization of (2.1) should come as no surprise. After all, for any finite horizon firm with horizon T, formula (2.3) defines the dividend approach where $V(T)$ denotes the total liquidating dividend in period T. The value $V(T+1)$ of the firm in period $(T+1)$ is identically zero in this case. The natural extension of these two ideas for an infinite horizon is clearly (3.3) and the transversality condition (1.2), and *not* (2.1) and (1.2) as in MM.

Sethi, Derzko, and Lehoczky (1992) even go further and show that the traditional dividend stream approach defined by equations (2.1), (1.2), and (1.5) does not have any positive solution $V(t)$, $p(t)$ and $n(t)$ that satisfies all three equations if

$$\sum_{s=0}^{\infty}\left[\frac{D(s)}{\sum_{\tau=s}^{\infty}\{D(\tau)-E(\tau)\}}\right]<\infty.$$

4. Concluding Remarks

In this paper, we have extended the MM framework to allow for firms that may purchase its own shares. We show that the traditional dividend stream approach as formulated in MM breaks down for a class of firms. Such a class was completely characterized in Sethi et al. (1982, 1991) and Sethi (1996). We give a modified dividend stream approach as a natural generalization of the dividend stream approach for finite horizon firms. We prove that the modified dividend stream approach is equivalent to the fundamental valuation approach. Furthermore, in Sethi et al. (1991, 1996) and Sethi et al. (1984) carried out in the presence of uncertainty, the ideas and the results of this paper have been extended to general stochastic environments in discrete time and continuous time, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests.

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