Available online at http://scik.org
Math. Finance Lett. 2018, 2018:1
https://doi.org/10.28919/mfl/3587
ISSN: 2051-2929

TERM STRUCTURE OF INTEREST RATES WITH STICKINESS: A SUBDIFFUSION APPROACH

ZHIGANG TONG[1,2,*], ALLEN LIU[2]

[1]Department of Mathematics and Statistics, University of Ottawa, 585 King Edward, Ottawa, Ontario, K1N 6N5, Canada

[2]Enterprise Risk and Portfolio Management, Bank of Montreal, 27th Floor, First Canadian Place, Toronto, Ontario, M5X 1A3, Canada

Abstract. In this paper, we propose a new class of term structure of interest rate models which is built on the subdiffusion processes. We assume that the spot rate is a function of a time changed diffusion process belonging to a symmetric pricing semigroup for which its spectral representation is known. The time change process is taken to be an inverse Lévy subordinator in order to capture the stickiness feature observed in the short-term interest rates. We derive the analytical formulas for both bond and bond option prices based on eigenfunction expansion method. We also numerically implement a specific subdiffusive model by testing the sensitivities of bond and bond option prices with respect to the parameters of time change process.

Keywords: subdiffusion; term structure of interest rate; option pricing; inverse Lévy subordinator; eigenfunction expansion

2010 AMS Subject Classification: 91G20, 91G30, 91G80.

*Corresponding author

E-mail address: ztong043@uottawa.ca

Received November 23, 2017

1. Introduction

The term structure of interest rates plays a significant role in financial markets. It can be applied in the valuation and hedging of interest rate contingent claims such as bonds and bond options. There is a vast literature of approaches to term structure modeling. In this paper we will focus on the spot rate diffusion models where the instantaneous interest rate (spot rate) is modeled as a function of a diffusion process.

In the pioneering work by Vasicek [23], an Ornstein Uhlenbeck (OU) process is proposed for spot rate. Cox, Ingersoll and Ross (CIR) [5] propose the square root process for the spot rate in a general equilibrium framework in order to introduce heteroscedasticity in the spot rate dynamics. Duffie and Kan [6] generalize the Vasicek model and CIR model and propose the most general affine class model, where both the mean and variance of spot rate are modeled as linear functions of spot rate. Chan et. al. [4], Aït-Sahalia [2] and Santon [21] perform the empirical comparisons among different models and find that the mean and variance of spot rates are nonlinear. To capture the nonlinearity, several alternative models to affine models have been proposed. Among all those models, the 3/2 model of Ahn and Gao [1] and the quadratic model of Leippold and Wu [10] are important ones. All of these popular classical models share a number of desirable features. First, they all include mean reversion in the underlying processes. Second, they provide closed-form expressions for transition and marginal densities of interest rate. Third, they are analytically tractable and easy to implement.

Despite the attractive properties of the classical models, no model has managed to capture all the characteristics of the qualitative behavior of interest rates. One of the striking features of short-term interest rate, especially the policy rate, is the change in the rate occurs only on pre-determined monetary-policy meeting dates and is in magnitude of discrete basis points. This infrequent and discrete change in the short rate or so-called stickiness of interest rate becomes more obvious after the global financial crisis in 2008. For most of developed countries, the policy rates drop to zero lower bound (ZLB) and tend to stay approximately static around ZLB for an extended period of time. The classical models have difficulties in accommodating this stickiness property because mean-reversion always pulls back the rates to their long-horizon values once the rates approach ZLB. Therefore, the occurrence of sustained periods of near-zero interest rate is very improbable.

To capture the stickiness in short rate, in this paper we present a new class of term structure models that is built on subdiffusive processes. The spot rate is modeled as the composition of two independent processes. The first process is a continuous function of a general time-homogeneous diffusion process belonging to a symmetric pricing semigroup for which its spectral representation is known. Most of the popular processes in finance, such as OU, CIR and 3/2 processes, satisfy this property. The second process is an inverse subordinator of a general infinite divisible distribution. The resulting interest rate process can exhibit constant time periods or so-called trapping events and the length of constant periods follows the probability law of the specified Lévy subordinator.

In finance, Magdziarz [16] generalizes the Black-Scholes model by a subdiffusive geometric Brownian motion as a model of stock prices exhibiting subdiffusive dynamics. Magdziarz et. al. [17] also consider a subdiffusive Brownian motion as a model of stock price. Karipova and Magdziarz [9] extend the multidimensional Black-Scholes model to the subdiffusive case to derive the pricing formula for basket options written on stock prices. Janczura et. al. [8] propose a subdiffusive OU model as a tool for financial data description. They also develop the parameter estimation method and calibrate the model to the interest rate data. Gajda and Wyłomańska [7] extend [8] by considering so-called time changed OU process, in which time is replaced by an inverse subordinator of general infinite divisible distribution.

Our main contributions in this paper are three-fold. First, we extend the works of [7] to more general processes and indeed, subdiffusive OU model is a special case of the models we consider. Second, we demonstrate how to price interest rate contingent claims under subdiffusive processes and derive the closed-form solutions to bond and bond option prices. Third, the technique that enables us to obtain the analytical formulas for interest rate products is eigenfunction expansion. The application of eigenfunction expansion method to time changed processes in finance can be found in [18, 13, 19, 12, 11]. However, in all of these applications, the time change processes are assumed to be either Lévy subordinators or integral of some absolutely continuous processes. A novelty of this paper is to extend the application of eigenfunction expansion method to subdiffusive processes.

The structure of the paper is as follows. Section 2 introduces the spot rate models for both diffusion and subdiffusion cases. For the diffusion model, the spectral representation of the underlying process is described. Several examples with known eigenfunctions and eigenvalues are provided. Then the inverse Lévy subordinator for the subdiffusion models is introduced. The Laplace transform of the inverse Lévy subordinator is provided for some important examples. Section 3 derives the closed-form formulas for both bond and bond option prices in terms of eigenfunction expansion. Section 4 numerically explores the sensitivities of the bond and bond option prices with respect to the parameters of the time change process for a specific model.

2. Spot rate models

2.1. **Spot rate diffusion models.** Let $(\Omega, \mathscr{F}, Q)$ denote a probability space with an information filtration $(\mathscr{F}_t)$. We assume that under the risk-neutral measure Q, the spot rate $r(t)$ is a function of the state variable $X(t)$, i.e. $r(t) = g(X(t))$, where $X(t)$ is a one-dimensional time-homogeneous diffusion process on an interval $I \subseteq \mathbb{R}$ with left and right endpoints l and r. The dynamics of $X(t)$ is

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t) \,, \tag{2.1}$$

with drift and diffusion terms denoted by μ and σ, respectively. We assume

Assumption 2.1. *The diffusion X is conservative, that is, $P_t(x,I) = 1$ for $t \geq 0$ and $x \in I$, where $P_t(x,A)$ is the transition function from initial state x to the Borel set $A \subseteq I$ in time t.*

Assumption 2.2. *$\mu(x)$, $\sigma(x) > 0$, $g(x)$ are continuous functions on open interval (l,r).*

Define $s(x)$ and $m(x)$ to be the scale and speed densities of the diffusion process X, respectively,

$$s(x) = \exp\left(-\int_{\varepsilon}^{x} \frac{2\mu(y)}{\sigma^2(y)} dy \right), \quad m(x) = \frac{2}{\sigma^2(x)s(x)} \,,$$

where $\varepsilon \in I$ is an arbitrary point. We further assume

Assumption 2.3.

$$\int_l^{\varepsilon} m(x)dx < \infty, \quad \int_{\varepsilon}^{r} m(x)dx < \infty \,.$$

.

Under Assumption 2.3, the endpoints of X can be regular, entrance or non-attracting natural boundaries according to Feller's boundary classification criteria (see e.g. [3] for more details). An endpoint is unattainable if it is a natural or an entrance boundary and is attainable if it is a regular boundary. In this paper, we assume

Assumption 2.4. *The infinite boundaries are unattainable and the regular boundaries instantaneously reflecting.*

Under this assumption, the regular reflecting boundaries are included in the state space I but the entrance and natural boundaries are not included.

Now consider the family of pricing operators (Feynman-Kac (FK) operators) defined by

$$\mathscr{P}_t f(x) = \mathrm{E}\left[\exp\left(-\int_0^{\mathrm{t}} \mathrm{g(X(u))du}\right)\mathrm{f(X(t))|X(0)=x}\right]. \tag{2.2}$$

Remark 2.5. The FK semigroup $(\mathscr{P}_t)_{t\geq 0}$ can be turned into the transition semigroup of another process $\hat{X}(t)$ (see e.g. [13, 15]):

$$\hat{X}(t) := \begin{cases} X(t), & \int_0^t g(X(u)du < E \\ \partial, & \int_0^t g(X(u)du \geq E \end{cases} \quad , \tag{2.3}$$

where E is an exponential random variable with parameter 1. $\hat{X}$ is called the process obtained from X by killing with respect to the positive continuous additive functional $A(t) = \int_0^t g(X(u))du$. $(\mathscr{P}_t)_{t\geq 0}$ is the transition semigroup of $\hat{X}$, i.e.

$$\mathscr{P}_t f(x) = \mathrm{E}[\mathrm{f}(\hat{\mathrm{X}}(\mathrm{t}))1_{\{\xi > \mathrm{t}\}}|\mathrm{X}(0) = \mathrm{x}]\,, \tag{2.4}$$

where $\xi := \inf\{t \geq 0 : \hat{X}(t) = \partial\}$ and by convention, $f(\partial) = 0$.

Under the Assumptions 2.1-2.4, these operators form a strongly continuous semigroup on the Banach space of bounded continuous functions on I with the supremum norm (see e.g. [3]).

The infinitesimal generator $\mathscr{L}$ of this semigroup is defined by

$$\mathscr{L}f(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) - g(x)f(x) \,, \tag{2.5}$$

where f is a transformation function. f' and f'' are first- and second-order derivatives of f, respectively.

Under our assumptions, the infinitesimal generator $\mathscr{L}$ with domain dom($\mathscr{L}$) is always self-adjoint on the Hilbert space $L^2(I,m)$ of functions on I square integrable with the speed measure $m(dx) = m(x)dx$ and endowed with the inner product

$$(f,g) = \int_I f(x)g(x)m(x)dx \,.$$

Thus, the Spectral Theorem for self-adjoint operators in Hilbert spaces can be applied to write down the spectral decomposition of the generator. In this paper, we further assume

Assumption 2.6. *The spectrum of the negative of the infinitesimal generator $-\mathscr{L}$ is purely discrete.*

Then, for any $f \in L^2(I,m)$, we have the following result (see e.g. [13, 15, 14]):

Theorem 2.7. *Under Assumptions 2.1-2.6, the FK semigroup $(\mathscr{P}_t)_{t\geq 0}$ on $L^2(I,m)$ and its infinitesimal generator $\mathscr{L}$ have spectra with eigenvalues* $\exp(-\lambda_n t)_{n\in\mathbb{N}}$ *and $(-\lambda_n)_{n\in\mathbb{N}}$, respectively. $\mathscr{P}_t f(x)$ has an eigenfunction expansion of the form:*

$$\mathscr{P}_t f(x) = \sum_{n=0}^{\infty} f_n \exp(-\lambda_n t)\,\psi_n(x) \,, \tag{2.6}$$

where $f_n = (f,\psi_n)$, $\{\lambda_n\}$ are the eigenvalues of $-\mathscr{L}$ and $\{\psi_n\}$ are the corresponding eigenfunctions satisfying the following Sturm-Liouville equation

$$-\mathscr{L}\psi_n = \lambda_n \psi_n \,. \tag{2.7}$$

For most of important term structure models, our Assumptions 2.1-2.6 are satisfied and the eigenvalues and eigenfunctions can be expressed in closed form (see [13, 15, 14] for more details).

Example 2.8. Vasicek model. $r(t) = X(t)$ and

$$dX(t) = \kappa(\theta - X(t))dt + \sigma dB(t) ,$$

where $\kappa > 0$, $\theta > 0$ and $\sigma > 0$.

The eigenvalues λ_n, $n = 0, 1, \ldots$, are

$$\lambda_n = \theta + \kappa n - \frac{\sigma^2}{2\kappa^2} .$$

The eigenfunctions ψ_n, $n = 0, 1, \ldots$, can be written as

$$\psi_n(x) = N_n \exp\Big(- a\xi - \frac{a^2}{2}\Big) H_n(\xi + a) ,$$

where

$$\xi = \frac{\sqrt{\kappa}}{\sigma}(x - \theta), \;\; a = \frac{\sigma}{\kappa^{3/2}}, \;\; N_n = \sqrt{\sqrt{\frac{\kappa}{\pi}} \frac{\sigma}{2^{n+1} n!}} ,$$

and $H_n(x)$ is the Hermite polynomials defined as

$$H_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!}(2x)^{n-2m} .$$

Example 2.9. CIR model. $r(t) = X(t)$ and

$$dX(t) = \kappa(\theta - X(t))dt + \sigma \sqrt{X(t)} dB(t) ,$$

where $\kappa > 0$, $\theta > 0$, $\sigma > 0$ and the Feller's condition $2\kappa\theta > \sigma^2$ is satisfied.

The eigenvalues λ_n, $n = 0, 1, \ldots$, are

$$\lambda_n = \gamma n + \frac{b}{2}(\gamma - \kappa) ,$$

where

$$\gamma = \sqrt{\kappa^2 + 2\sigma^2}, \;\; b = \frac{2\kappa\theta}{\sigma^2} .$$

The eigenfunctions ψ_n, $n = 0, 1, \ldots$, can be written as

$$\psi_n(x) = N_n \exp\left(\frac{(\kappa - \gamma)x}{\sigma^2} \right) L_n^{(b-1)}\Big(\frac{2\gamma x}{\sigma^2} \Big) ,$$

where

$$N_n = \sqrt{\frac{\sigma^2 n!}{2\Gamma(b+n)}}\left(\frac{2\gamma}{\sigma^2}\right)^{b/2},$$

where $\Gamma(\cdot)$ is Gamma function and $L_n^{(v)}(x)$ is the generalized Laguerre polynomials defined as

$$L_n^{(v)}(x) = \frac{(v+1)_n}{n!}\,{}_1\mathrm{F}_1(-n; v+1; x),$$

where ${}_1\mathrm{F}_1(a; b; x)$ is the Kummer confluent hypergeometric function, given by

$${}_1\mathrm{F}_1(a; b; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{x^n}{n!},$$

where $(a)_0 = 1$, $(a)_n = a(a+1),\ldots,(a+n-1)$, $n > 0$.

Example 2.10. 3/2 model. $r(t) = X(t)$ and

$$dX(t) = \kappa(\theta - X(t))X(t)dt + \sigma X(t)^{3/2}dB(t),$$

where $\kappa > 0$, $\theta > 0$ and $\sigma > 0$.

The eigenvalues λ_n, $n = 0, 1, \ldots$, are

$$\lambda_n = \kappa\theta\left(n + \nu - \alpha + \frac{1}{2}\right),$$

where

$$\alpha = \frac{\kappa}{\sigma^2} + 1, \;\; \nu = \sqrt{\left(\frac{\kappa}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2}{\sigma^2}}.$$

The eigenfunctions ψ_n, $n = 0, 1, \ldots$, can be written as

$$\psi_n(x) = N_n x^{\alpha-\nu-1/2} L_n^{(2\nu)}\left(\frac{\beta}{x}\right),$$

where

$$\beta = \frac{2\kappa\theta}{\sigma^2}, \;\; N_n = \sqrt{\frac{\sigma^2 \beta^{2\nu+1} n!}{2\Gamma(2\nu + n + 1)}}.$$

2.2. **Spot rate subdiffusion models.** Although the classical diffusion models can capture some important features of interest rates such as mean reversion, they still have some limitations. The changes in interest rates, especially short-term rates, tend to occur infrequently. After the global financial crisis in 2008, the short-term rates for many countries stay at close to ZLB level for extended periods of time. This stickiness phenomenon of interest rates offers serious challenge to the classical models since they cannot produce sustained periods of constant interest rates. To capture the stickiness in interest rates, we extend the spot rate diffusion models to spot rate subdiffusion models.

Let $L(t)$ be a Lévy subordinator with the Laplace transform

$$\mathrm{E}[\exp(-\lambda \mathrm{L}(\mathrm{t}))] = \exp(-\mathrm{t}\phi(\lambda)) \ , \tag{2.8}$$

where ϕ is the Lévy exponent and given by the Lévy-Khintchine formula (see e.g. [22])

$$\phi(\lambda) = \gamma\lambda + \int_{(0,\infty)} (1 - \exp(-\lambda s))\nu(ds) \ , \tag{2.9}$$

where $\gamma \geq 0$ and the Lévy measure ν satisfies

$$\int_{(0,\infty)} (s \wedge 1)\nu(ds) < \infty \ .$$

Define $T(t)$ to be an inverse subordinator corresponding to $L(t)$, that is

$$T(t) = \inf\{\tau > 0 : L(\tau) > t\} \ . \tag{2.10}$$

To construct spot rate subdiffusion model, we first define a new process

$$\hat{X}^{\phi}(t) = \hat{X}(T(t)) \ , \tag{2.11}$$

where $\hat{X}$ is defined in (2.3).

For the process $\hat{X}^{\phi}$, define a new pricing operator

$$\mathscr{P}_t^{\phi} f(x) = \mathrm{E}\left[\mathrm{f}(\hat{\mathrm{X}}^{\phi}(\mathrm{t}))1_{\{\xi > \mathrm{T}(\mathrm{t})\}} | \mathrm{X}(0) = \mathrm{x}\right] \ , \tag{2.12}$$

where $\xi := \inf\{t \geq 0 : \hat{X}(t) = \partial\}$.

From (2.2) and (2.4), we know $\mathscr{P}_t^{\phi} f(x)$ can also be written as

$$\begin{aligned}\mathscr{P}_t^{\phi} f(x) &= \mathrm{E}\left[\exp\left(-\int_0^{\mathrm{T(t)}} \mathrm{g(X(u))du}\right)\mathrm{f(X(T(t)))|X(0)=x}\right]\\ &= \mathrm{E}\left[\exp\left(-\int_0^{\mathrm{t}} \mathrm{g}^{\phi}\mathrm{(Y(u))du}\right)\mathrm{f(Y(t))|Y(0)=x}\right],\end{aligned} \tag{2.13}$$

where $Y(t) = X(T(t))$ and $g^{\phi}(x)$ is the discount rate function.

In the subdiffusive model, the spot rate is

$$r(t) = g^{\phi}(Y(t)), \tag{2.14}$$

where

$$Y(t) = X(T(t)), \tag{2.15}$$

where $X(t)$ is the diffusion process defined in Section 2.1 and assumed to be independent of $T(t)$.

Therefore, the spot rate $r(t)$ is a time changed diffusion process where the time change process T is an inverse subordinator. Since the Lévy subordinator L is a nondecreasing process with positive jumps and nonnegative drift, the subdiffusive models can produce periods of constant values. The distribution of waiting times in which the rates do not change follows the probability law of L.

We can use the following theorem to compute $\mathscr{P}_t^{\phi} f(x)$ for any $f \in L^2(I,m)$:

Theorem 2.11. *For the subdiffusive process $r(t)$ defined in (2.14) and (2.15), under Assumptions 2.1-2.6, we have*

$$\mathscr{P}_t^{\phi} f(x) = \sum_{n=0}^{\infty} f_n \eta(\lambda_n, t)\psi_n(x), \tag{2.16}$$

where $f_n = (f, \psi_n)$. λ_n and ψ_n are the eigenvalues and eigenfunctions of infinitesimal generator $-\mathscr{L}$ corresponding to FK pricing operator $\mathscr{P}_t$. $\eta(u,t)$ is the Laplace transform of the inverse subordinator $T(t)$, namely

$$\eta(u,t) = \mathrm{E}[\exp(-\mathrm{uT(t)})]. \tag{2.17}$$

Proof. Using the law of iterated expectations and Theorem 2.7, we have

$$
\begin{aligned}
& \mathrm{E}\left[\exp\left(-\int_0^{\mathrm{t}} \mathrm{g}^{\phi}(\mathrm{Y(u)})\mathrm{du}\right)\mathrm{f(Y(t))}|\mathrm{Y(0)}=\mathrm{x}\right] \\
&= \mathrm{E}[\mathrm{f}(\hat{\mathrm{X}}^{\phi}(\mathrm{t}))1_{\{\xi>\mathrm{T(t)}\}}|\mathrm{X(0)}=\mathrm{x}] \\
&= \mathrm{E}[\mathrm{f}(\hat{\mathrm{X}}(\mathrm{T(t)}))1_{\{\xi>\mathrm{T(t)}\}}|\mathrm{X(0)}=\mathrm{x}] \\
&= \mathrm{E}\left\{\mathrm{E}[\mathrm{f}(\hat{\mathrm{X}}(\mathrm{T(t)}))1_{\{\xi>\mathrm{T(t)}\}}|\mathrm{X(0)}=\mathrm{x},\mathrm{T(t)}]\right\} \\
&= \mathrm{E}\left[\sum_{\mathrm{n}=0}^{\infty}\mathrm{f_n}\exp(-\lambda_{\mathrm{n}}\mathrm{T(t)})\psi_{\mathrm{n}}(\mathrm{x})\right] \\
&= \sum_{n=0}^{\infty} f_n \eta(\lambda_n,t)\psi_n(x)\,.
\end{aligned}
$$

□

From Theorem 2.11, it is clear that the eigenfunction expansion of subdiffusion process has the same form as diffusion process, but with $\exp(-\lambda_n t)$ replaced by the Laplace transform of the inverse subordinator $T(t)$. This explains why the eigenfunction expansion method is well suited for computing the contingent claim prices with the time-changed process.

To compute $\eta(u,t)$, we can first use the following lemma to calculate the Laplace transform $\hat{\eta}(u,k)$ of $\eta(u,t)$.

Lemma 2.12. *The Laplace transform $\hat{\eta}(u,k)$ of $\eta(u,t)$ is*

$$
\hat{\eta}(u,k) = \int_0^{\infty}\exp(-kt)\eta(u,t)dt = \frac{\phi(k)}{k(u+\phi(k))}\,, \tag{2.18}
$$

where ϕ is the Lévy exponent defined in (2.9).

Proof. Let $f(t,x)$ be the PDF of $T(t)$ and $g(\tau,x)$ be the PDF of $L(\tau)$. From $P(T(t)\le x) = P(L(x)>t)$, we have

$$
f(t,x) = -\frac{\partial}{\partial x}\int_0^t g(x,s)ds\,.
$$

Then, we get

$$
\hat{f}(k,x) = \int_0^{\infty}\exp(-kt)f(t,x)dt = \frac{\phi(k)}{k}\exp(-x\phi(k))\,.
$$

Then,

$$
\begin{aligned}
\hat{\eta}(u,k) &= \int_0^\infty \exp(-kt)\eta(u,t)dt \\
&= \int_0^\infty \exp(-kt)\int_0^\infty \exp(-ux)f(t,x)dxdt \\
&= \int_0^\infty \exp(-ux)\int_0^\infty \exp(-kx)f(t,x)dtdx \\
&= \frac{\phi(k)}{k}\int_0^\infty \exp(-(u+\phi(k))x)dx \\
&= \frac{\phi(k)}{k(u+\phi(k))} \, .
\end{aligned}
$$

□

Given the Laplace transform of $\eta(u,t)$, we can obtain $\eta(u,t)$ by taking inverse Laplace transform. For certain Lévy subordinator L, we can obtain the closed-form solution to $\eta(u,t)$ (see e.g. [7] for detailed derivations).

Example 2.13. Inverse α-stable subordinator. Let $L(t)$ be a stable subordinator with Lévy exponent $\phi(k) = k^\alpha$ with $\alpha \in (0,1)$. Then, $\eta(u,t)$ for $T(t)$ is

$$
\eta(u,t) = E^1_{\alpha,1}(-ut^\alpha) \, ,
$$

where $E^\gamma_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function:

$$
E^\gamma_{\alpha,\beta}(z) = \sum_{j=0}^\infty \frac{(\gamma)_j z^j}{j!\Gamma(\alpha j+\beta)} \, ,
$$

where α, β, $\gamma \in \mathbb{C}$, $\mathbb{R}\mathrm{e}(\alpha)$, $\mathbb{R}\mathrm{e}(\beta)$, $\mathbb{R}\mathrm{e}(\gamma) > 0$.

Example 2.14. Inverse transient subordinator. Let $L(t)$ be a transient subordinator with Lévy exponent $\phi(k) = c_1k^{\alpha_1} + c_2k^{\alpha_2}$ with α_1, $\alpha_2 \in (0,1)$, $\alpha_1 < \alpha_2$, c_1, $c_2 \geq 0$ and $c_1 + c_2 = 1$. Then, we have

$$
\begin{aligned}
\eta(u,t) = &\sum_{j=0}^\infty \left(-\frac{c_1 t^{\alpha_2-\alpha_1}}{c_2}\right)^j E^{j+1}_{\alpha_2,(\alpha_2-\alpha_1)j+1}\left(-\frac{ut^{\alpha_2}}{c_2}\right) \\
&- \sum_{j=0}^\infty \left(-\frac{c_1 t^{\alpha_2-\alpha_1}}{c_2}\right)^{j+1} E^{j+1}_{\alpha_2,(\alpha_2-\alpha_1)(j+1)+1}\left(-\frac{ut^{\alpha_2}}{c_2}\right) \, .
\end{aligned}
$$

Example 2.15. Inverse tempered α-stable subordinator. Let $L(t)$ be a tempered α-stable subordinator with Lévy exponent $\phi(k) = (k+\vartheta)^\alpha - \vartheta^\alpha$ with $\vartheta > 0$ and $\alpha \in (0,1)$. Then, we have

$$\eta(u,t) = 1 - u\int_0^t \exp(-\vartheta\tau)\tau^{\alpha-1}E^1_{\alpha,\alpha}((\vartheta^\alpha - u)\tau^\alpha)d\tau \, .$$

3. Bond and bond option pricing in subdiffusion models

Having obtained the eigenfunction expansion of pricing operators for subdiffusive models, we can derive the analytical solutions to the prices of interest rate contingent claims. In this section, we will focus on pricing of two important fixed-income products: bonds and bond options. To save space, we only give the pricing formulas for the subdiffusive 3/2 model and we emphasize that our technique can be easily applied to other models, such as subdiffusive OU or CIR models. In the subdiffusive 3/2 model, $g(X(t)) = X(t)$ and

$$r(t) = g^\phi(Y(t)) = g^\phi(X(T(t))) \, , \tag{3.1}$$

$$dX(t) = \kappa(\theta - X(t))X(t)dt + \sigma X(t)^{3/2}dB(t) \, , \tag{3.2}$$

where $\kappa > 0$, $\theta > 0$ and $\sigma > 0$. $g^\phi(x)$ and $g(x)$ are related via (2.13).

In the 3/2 spot diffusion model of [1], the drift and diffusion of spot rate are nonlinear functions of spot rate. The authors of [1] show that in empirical analysis, the 3/2 model outperforms the affine models in both time-series as well as cross-sectional tests. For the subdiffusive 3/2 model, the eigenvalues and eigenfunctions can be computed in closed form as in Example 2.10. The only thing left for the calculation of bond and bond option prices is to compute the eigenfunction expansion coefficient f_n.

Let $B(t,T,y(t))$ represent the time t price of zero coupon bond that matures at time T given $Y(t) = y(t)$. The bond prices can be computed from the following theorem.

Theorem 3.1. *For the subdiffusive 3/2 model specified in (3.1) and (3.2), the zero coupon bond price $B(t,T,y(t))$ is given by*

$$B(t,T,y(t)) = \sum_{n=0}^{\infty} c_n \eta(\lambda_n, T-t)\psi_n(y(t)) \, , \tag{3.3}$$

where

$$c_n = \frac{2N_n}{\sigma^2}\beta^{-\alpha-\nu-1/2}a_n(\alpha+\nu+\frac{1}{2},1,2\nu)\ ,$$

where α*,* β*,* ν *and* N_n *are given in Example 2.10 and the function* $\eta(u,t)$ *is given in (2.17). In addition,*

$$a_n(\delta,c,\lambda) = \frac{(1-\delta+\lambda)_n}{n!c^\delta}\Gamma(\delta)\ .$$

Proof. From Theorem 2.11, we know the eigenfunction expansion for bond price is

$$\begin{aligned} B(t,T,y(t)) &= \mathrm{E}\left[\exp\left(-\int_{\mathrm{t}}^{\mathrm{T}} \mathrm{g}^\phi(\mathrm{Y(u)})\mathrm{du}\right)|\mathrm{Y(t)} = \mathrm{y(t)}\right] \\ &= \sum_{n=0}^{\infty} c_n\eta(\lambda_n,T-t)\psi_n(y(t))\ , \end{aligned}$$

where

$$c_n = \int_0^\infty \psi_n(x)m(x)dx\ ,$$

and the speed density function $m(x)$ for the 3/2 process is given by

$$m(x) = \frac{2}{\sigma^2}x^{-2\alpha-1}\exp\Big(-\frac{\beta}{x}\Big)\ .$$

Therefore,

$$\begin{aligned} c_n &= \int_0^\infty N_n x^{-\alpha-\nu-1/2}L_n^{(2\nu)}\Big(\frac{\beta}{x}\Big)\frac{2}{\sigma^2}x^{-2\alpha-1}\exp\Big(-\frac{\beta}{x}\Big)dx \\ &= \frac{2N_n}{\sigma^2}\beta^{-\alpha-\nu-1/2}\int_0^\infty x^{\alpha+\nu-1/2}\exp(-x)L_n^{(2\nu)}(x)dx \\ &= \frac{2N_n}{\sigma^2}\beta^{-\alpha-\nu-1/2}a_n(\alpha+\nu+\frac{1}{2},1,2\nu)\ , \end{aligned} \tag{3.4}$$

where we have used the formula (see e.g. [20])

$$\int_0^\infty x^{\delta-1}\exp(-cx)L_n^{(\lambda)}(cx)dx = \frac{(1-\delta+\lambda)_n}{n!c^\delta}\Gamma(\delta) := a_n(\delta,c,\lambda)\ .$$

□

Let $P(t,K,T,T_1,y(t))$ be the time t price of a put option on a zero coupon bond with strike price K and maturity T, where the bond matures at time T_1, given $Y(t) = y(t)$. We can use the following theorem to compute the bond option price.

Theorem 3.2. *For the subdiffusive 3/2 model specified in (3.1) and (3.2), the bond option price $P(t,K,T,T_1,y(t))$ is given by*

$$P(t,K,T,T_1,y(t)) = K\sum_{n=0}^{\infty} d_n \eta(\lambda_n, T-t)\psi_n(y(t)) - \sum_{n=0}^{\infty} f_n \eta(\lambda_n, T-t)\psi_n(y(t)) \,, \tag{3.5}$$

where

$$d_n = \frac{2N_n}{\sigma^2}\beta^{-\alpha-\nu-1/2} b_n\Big(\frac{\beta}{y^*}, \alpha+\nu-\frac{1}{2}, 1, 2\nu\Big) \,, \tag{3.6}$$

where the function $\eta(u,t)$ is given in (2.17). y^* is the solution to the equation $K = B(T,T_1,y)$ and

$$\begin{aligned} f_n = &\sum_{m=0}^{\infty} c_m \eta(\lambda_m, T_1 - T)\frac{2N_n N_m}{\sigma^2}\beta^{-2\nu-1}\frac{\Gamma(2\nu+m+1)\Gamma(2\nu+n+1)}{\Gamma(2\nu+m+n+1)} \\ &\times \sum_{l=0}^{\min(m,n)} \frac{(m+n-2l)!}{l!(m-l)!(n-l)!\Gamma(2\nu+l+1)} b_{m+n-2l}\Big(\frac{\beta}{y^*}, 2\nu+2l, 1, 2\nu+2l\Big) \,, \end{aligned}$$

where c_m is given in (3.4) and

$$b_n(x,\delta,a,\lambda) = \frac{(\lambda+1)_n}{n!(\delta+1)} x^{\delta+1} {}_2F_2(\lambda+n+1, \delta+1; \lambda+1, \delta+2; -ax) \,,$$

where ${}_2F_2(a,b;c,d;x)$ is the hypergeometric function defined by

$${}_2F_2(a,b;c,d;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (d)_n}\frac{x^n}{n!} \,.$$

Proof. From Theorem 2.11, we know the eigenfunction expansion for bond option price is

$$\begin{aligned} P(t,K,T,T_1,y(t)) &= \mathrm{E}\bigg[\exp\bigg(-\int_{\mathrm{t}}^{\mathrm{T}} \mathrm{g}^{\phi}(\mathrm{Y(s)})\mathrm{ds}\bigg)(\mathrm{K}-\mathrm{B(T,T_1,Y(T))})^{+}|\mathrm{Y(t)=y(t)}\bigg] \\ &= K\mathrm{E}\bigg[\exp\bigg(-\int_{\mathrm{t}}^{\mathrm{T}} \mathrm{g}^{\phi}(\mathrm{Y(s)})\mathrm{ds}\bigg)1_{\{\mathrm{Y(T)}>\mathrm{y}*\}}|\mathrm{Y(t)=y(t)}\bigg] \\ &\quad -\mathrm{E}\bigg[\exp\bigg(-\int_{\mathrm{t}}^{\mathrm{T}} \mathrm{g}^{\phi}(\mathrm{Y(s)})\mathrm{ds}\bigg)\mathrm{B(T,T_1,Y(T))}1_{\{\mathrm{Y(T)}>\mathrm{y}*\}}|\mathrm{Y(t)=y(t)}\bigg] \\ &= K\sum_{n=0}^{\infty} d_n \eta(\lambda_n, T-t)\psi_n(y(t)) - \sum_{n=0}^{\infty} f_n \eta(\lambda_n, T-t)\psi_n(y(t)) \,, \end{aligned}$$

where y^* is the solution to the equation $K = B(T, T_1, y)$ and

$$\begin{aligned}
d_n &= \int_{y^*}^{\infty} \psi_n(x) m(x) dx \\
&= \frac{2N_n}{\sigma^2} \int_{y^*}^{\infty} x^{-\alpha-\nu-3/2} \exp\Big(-\frac{\beta}{x}\Big) L_n^{(2\nu)}\Big(\frac{\beta}{x}\Big) dx \\
&= \frac{2N_n}{\sigma^2} \beta^{-\alpha-\nu-1/2} \int_0^{\beta/y^*} y^{\alpha+\nu-1/2} \exp(-y) L_n^{(2\nu)}(y) dy \\
&= \frac{2N_n}{\sigma^2} \beta^{-\alpha-\nu-1/2} b_n\Big(\frac{\beta}{y^*}, \alpha+\nu-\frac{1}{2}, 1, 2\nu\Big) ,
\end{aligned}$$

where we use the fact (see e.g. [20])

$$\begin{aligned}
&\int_0^y x^{\delta} \exp(-ax) L_n^{(\lambda)}(ax) dx = \frac{(\lambda+1)_n}{n!(\delta+1)} x^{\delta+1} {}_2\mathrm{F}_2(\lambda+n+1, \delta+1; \lambda+1, \delta+2; -ax) \\
&:= b_n(x, \delta, a, \lambda) .
\end{aligned}$$

For f_n, we have

$$\begin{aligned}
f_n &= \sum_{m=0}^{\infty} c_m \eta(\lambda_m, T_1 - T) \int_{y^*}^{\infty} \psi_n(x) \psi_m(x) m(x) dx \\
&= \sum_{m=0}^{\infty} c_m \eta(\lambda_m, T_1 - T) \frac{2N_n N_m}{\sigma^2} \beta^{-2\nu-1} \int_0^{\beta/y^*} y^{2\nu} \exp(-y) L_n^{(2\nu)}(y) L_m^{(2\nu)}(y) dy .
\end{aligned}$$

Using the formula (see e.g. [20])

$$L_m^{(\delta)} L_n^{(\delta)} = \frac{\Gamma(\delta+m+1)\Gamma(\delta+n+1)}{\Gamma(\delta+m+n+1)} \sum_{l=0}^{\min(n,m)} \frac{(m+n-2l)!}{l!(m-l)!(n-l)!\Gamma(\delta+l+1)} x^{2l} L_{m+n-2l}^{(\delta+2l)}(x) ,$$

we have

$$\begin{aligned}
f_n = &\sum_{m=0}^{\infty} c_m \eta(\lambda_m, T_1 - T) \frac{2N_n N_m}{\sigma^2} \beta^{-2\nu-1} \frac{\Gamma(2\nu+m+1)\Gamma(2\nu+n+1)}{\Gamma(2\nu+m+n+1)} \\
&\times \sum_{l=0}^{\min(m,n)} \frac{(m+n-2l)!}{l!(m-l)!(n-l)!\Gamma(2\nu+l+1)} \int_0^{\beta/y^*} y^{2\nu+2l} \exp(-y) L_{m+n-2l}^{(2\nu+2l)}(y) dy \\
= &\sum_{m=0}^{\infty} c_m \eta(\lambda_m, T_1 - T) \frac{2N_n N_m}{\sigma^2} \beta^{-2\nu-1} \frac{\Gamma(2\nu+m+1)\Gamma(2\nu+n+1)}{\Gamma(2\nu+m+n+1)} \\
&\times \sum_{l=0}^{\min(m,n)} \frac{(m+n-2l)!}{l!(m-l)!(n-l)!\Gamma(2\nu+l+1)} b_{m+n-2l}\Big(\frac{\beta}{y^*}, 2\nu+2l, 1, 2\nu+2l\Big) .
\end{aligned}$$

□

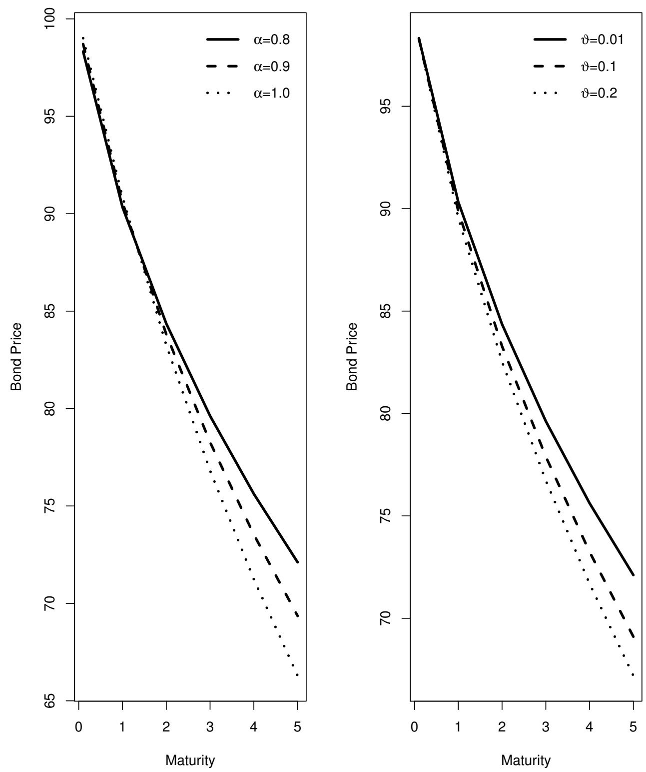

FIGURE 1. Bond prices for different α and ϑ. The parameters are $\kappa = 3.64$, $\theta = 0.08$, $\sigma = 1.28$ and $X(0) = 0.1$. In addition, in the left panel, $\vartheta = 0.01$ and in the right panel, $\alpha = 0.8$. The notional of the bond is 100.

4. Numerical experiment

In this section, we numerically study the subdiffusive 3/2 model, which is specified in Section 3. We choose inverse tempered α-stable subordinator in Example 2.15 as the time change process $T(t)$. The parameters for the classical 3/2 model are obtained from [1]. To compute the bond and bond option prices, we need to truncate the eigenfunction expansion after a finite number of terms. We follow [11] and truncate the infinite series when a given error tolerance level is reached. In practice, we find the convergence of the expansion is really fast.

In Fig. 1, we demonstrate the sensitivities of bond prices with respect to the parameters α and ϑ for T(t). First, we find that the bond prices are very sensitive to both parameters. Second, the bond price is an increasing function of α for the short-term bond but turns into a decreasing

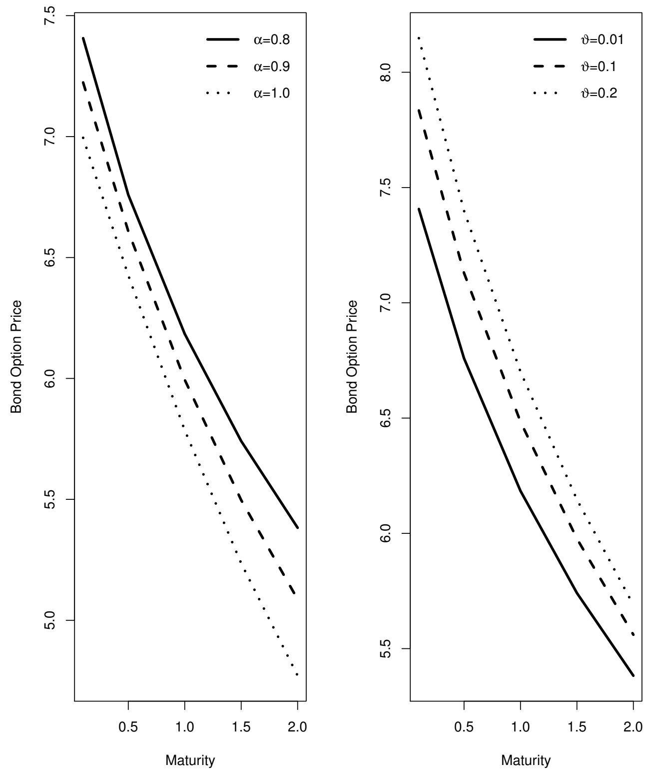

FIGURE 2. Bond option prices for different α and ϑ. The parameters are $\kappa = 3.64$, $\theta = 0.08$, $\sigma = 1.28$, $X(0) = 0.1$ and $K = 100$. In addition, in the left panel, $\vartheta = 0.01$ and in the right panel, $\alpha = 0.8$. The notional of the bond is 100.

function when maturities increase. Third, the bond price is a deceasing function of ϑ across all the maturities.

In Fig. 2, we also plot the term structure of bond option prices for various values of α and ϑ. Again, the bond option prices are very sensitive to both parameters. For the put options we consider, we find that the option price decreases with α but increases with ϑ.

5. Conclusion

The classical term structure models such as affine or non-linear models can capture the main characteristics of interest rates such as mean reversion. However, all the classical models have difficulties in producing the stickiness feature of short-term rates where the rates tend to stay unchanged for an extended period of time. To model the trapping event observed in the fixed-income markets, we propose a new class of term structure models which is built on the subdiffusion processes. In its most general form, the underlying process is a time-homogeneous diffusion process belonging to a symmetric pricing semigroup and also including mean-reversion. Then the process is time changed by an inverse Lévy subordinator. Using the eigenfunction expansion technique, we are able to derive the closed-form solutions to both bond and bond option prices. We consider several examples where the underlying processes have known spectral representation and the inverse subordinators allow for explicit Laplace transforms. We also numerically implement a specific subdiffusive model and explore its sensitives to the parameters of the time change process.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] D.H. Ahn, B. Gao, A parametric nonlinear model of term structure dynamics, Rev. Financ. Stud. 12 (1999) 721-762.

[2] Y. Aït-Sahalia, Nonparametric pricing of interest rate derivative securities, Econometrica, 64 (1996) 527-560.

[3] A.N. Borodin, P. Salminen, Handbook of Brownian Motion, second edition, Birkhäuser, 2002.

[4] K.C. Chan, G.A. Karolyi, F.A. Longstaff, A.B. Sanders, An empirical comparison of alternative models of short-term interest rate, J. Finance, 47 (1992) 1209-1228.

[5] J.C. Cox, J.E. Ingersoll, S.A. Ross, A theory of the term structure of interest rates, Econometrica, 53 (1985) 385-407.

[6] D. Duffie, R. Kan, A yield-factor model of interest rates, Math. Finance, 6 (1996) 379-406.

[7] J. Gajda, A. Wyłomańska, Time-changed Ornstein-Uhlenbeck process, J. Phys. A, 48 (2015), Article ID 135004.

[8] J. Janczura, S. Orzeł, A. Wyłomańska, Subordinated α-stable Ornstein-Uhlenbeck process as a tool for financial data description, Physica A, 390 (2011) 4379-4387.

[9] G. Karipova, M. Magdziarz, Pricing of basket options in subdiffusive fractional Black-Scholes model, Chaos Solitons Fractals, 102 (2017), 245-253.

[10] M. Leippold, L. Wu, Design and estimation of multi-currency quadratic models, Rev. Finance, 11 (2007) 167-207.

[11] J. Li, L. Li, G. Zhang, Pure jump models for pricing and hedging VIX derivatives, J. Econ. Dyn. Control, 74 (2017) 28-55.

[12] L. Li, V. Linetsky, Time-changed Ornstein-Uhlenbeck processes and their applications in commodity derivative models, Math. Finance, 24 (2014) 289-330.

[13] D. Lim, L. Li, V. Linetsky, Evaluating callable and putable bonds: an eigenfunction expansion approach, J. Econ. Dyn. Control, 36 (2012) 1888-1908.

[14] V. Linetsky, The spectral decomposition of the option value, Int. J. Theor. Appl. Finance, 7 (2004) 337-384.

[15] V. Linetsky, D. Mitchell, Spectral methods in derivatives pricing, in: J.R. Birge, V. Linetsky (Eds.), Handbook of Financial Engineering, Elsevier: Amsterdam, 2008, pp. 223-299.

[16] M. Magdziarz, Black-Scholes Formula in subdiffusive regime, J. Stat. Phys. 136 (2009) 553-564.

[17] M. Magdziarz, S. Orzeł, A. Weron, Option pricing in subdiffusive Bachelier model, J. Stat. Phys. 145 (2011) 187-203.

[18] R. Mendoza-Arriaga, P. Carr, V. Linetsky, Time changed Markov processes in unified credit-equity modeling, Math. Finance, 20 (2010) 527-569.

[19] R. Mendoza-Arriaga, V. Linetsky, Time-changed CIR default intensities with two-sided mean-reverting jumps, Ann. Appl. Probab. 24 (2013) 811-856.

[20] A.P. Prudnikov, Y.A. Brychkov, O.I. Marichev, Integrals and Series, Vol. 2, Gordon and Breach Science Publishers, 1986.

[21] R. Santon, A nonparametric model of term structure dynamics and the market price of interest rate risk, J. Finance, 52 (1997) 1973-2002.

[22] K. Sato, Lévy Processes and Infinitely Divisible Distribution, UK: Cambridge, 1999.

[23] O. Vasicek, An equilibrium characterization of the term structure, J. Financ. Econ. 5 (1977) 177-188.