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LIFECYCLE OPTIMAL INVESTMENT POLICY FOR PENSION FUNDS WITH TRANSACTION COST

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Abstract: In this paper, lifecycle investment of a Constant Relative Risk Aversion (CRRA) investor as a representative of pension plan participants is investigated. The investor has a finite investment horizon and is subject to the proportional transaction costs and a constant rate of return. Attempt is made to maximize the investor's utility by trading between stock and money market account. A set of partial differential equations are derived and closed form solution proffered. The effects of the volatility of the risky asset are investigated and it shows that a zero value of the volatility resulted to the value function equals zero and its unit value with the drift parameter ξ equals the discount rate k, gave an indeterminate value for the value function. Precise conditions are obtained which determine the growth rate of the value function in the sell and buy regions.

Keywords: Lifecycle investment, optimal portfolio selection, pension funds, constant rate of return, transaction costs, CRRA

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1. Introduction

Merton [1] pioneered in applying continuous-time stochastic models to the study of financial markets. In the absence of transaction costs, he showed that an optimal

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investment problem can be formulated as a Hamilton-Jacobi-Bellman (HJB) equation that allows an explicit solution for a constant relative risk aversion (CRRA) investor. The corresponding optimal investment policy is to keep a constant fraction of total wealth in each asset during the whole investment period. To implement the policy, the investor would have to indulge in incessant trading which is completely unrealistic in the face of transaction costs and violates the conventional largely buy-and-hold investment strategy.

To overcome the shortages, Magil and Constantinides [2] introduced proportional transaction costs to Merton's model. They provided a fundamental insight, that there is a no-trading region in the presence of transaction costs and the no-trading region must be a wedge. But, their argument is heuristic at best. In terms of a rigorous mathematical analysis, Davis and Norman [3] showed that for an infinite horizon investment and consumption with transaction costs, the optimal policies are determined by the solution of a free boundary problem, where the free boundaries correspond to the optimal buying and selling policies. Relying on the concept of viscosity solutions to HJB equations, Shreve and Soner [4] fully characterized the infinite horizon optimal policies. Using a martingale approach, Cvitanic and Karatzas [5] proved the existence of an optimal solution to the portfolio optimization problem with transaction costs. Other existence results can be found in Bouchard[6], Guasoni [7] and Guasoni and Schachermayer [8].

Traditional economic models on optimal investment and consumption policy have been extended on many directions ever since Merton’s [1], [9]. In these benchmark models investors can trade asset continuously at any time without incurring any kind of costs. However, in the capital market, an asset is also featured by its liquidity in addition to the commonly used risk and return.

Working with models that incorporate liquidity considerations, in whatever shape, requires a thorough re-examination of mainstream theory of financial markets. There are various ways to model illiquidity in an optimal investment problem. In this paper, transaction cost is used as a proxy for illiquidity due to the following two advantages: first it allows for mathematical flexibility and tractability; secondly it enables references and comparisons with a large body of previous literature. Amihud and Mendelson [19] Acharya and Pederson [11] among others derived clientele effect and spread-return in asset pricing with bid-spread ask and a liquidity-adjusted Capital Asset Pricing Model (CAPM) respectively. Longstaff [12],[13]; Schwartz and Tebaldi [14],Davis and Norman [3], considered an infinite horizon maximization problem with intermediate consumption, while Chellathurai and Draviam [15] considered a finite horizon portfolio selection without intermediate consumption when fixed and/or proportional transaction costs are present. An efficient and tractable numerical algorithm to obtain the boundaries using binomial approximation was done by Gennotte and Jung [16]. Zeriphopoulou et al [17] solved the European option pricing problem with transaction costs taking advantage of convergence of discretization of the stock price. Similarly, Balduzzi and Lynch [18] discrete both time and state to numerically compute the optimal investment policy for an investor with finite horizon. Jang et al 19] found that in contrast to the standard literature, transaction costs can have a first order effect, and investor responds to changes in either regime by adjusting consumption and investment policies if the market conditions change over time.

On the other hand, Constantinides [20], [21] has shown that the optimal transaction policy is to maintain the ratio of the dollar amount invested in the riskless asset to that in the risky asset within a certain range, represented by the buy boundary and sell boundary. Consequently three regions are identified depending on the portfolio ratio: the no-transaction region, the buy region and the sell region.

In particular, a CRRA investor investigated as a representative of pension plan participants, who has a finite horizon and is subject to proportional transaction costs when trading stock as well as money market account. Mathematically these conditions are boiled down to a set of parabolic differential equations. In contrast to the infinite horizon model in which stationary solutions can be obtained, the value function and the corresponding two boundaries strongly depend on horizon. Liu and Loewinstein [22] solved the deterministic finite horizon problem by making use of the exponentially distributed horizon. They claimed that proportional transaction costs together with a finite horizon would imply a time-varying, largely buy-and-hold trading strategy.

In our case, a closed form solution of the partial differential equation is given. The effects of the volatility of the risky asset are investigated. We show that a zero value of the volatility resulted to the value function equals zero and its unity value with $\xi = k$ led to the value function becoming indeterminate

2. Lifecycle investment without transaction costs.

We consider the optimal investment policy as well as the consumption-savings decision from a lifecycle perspective. The lifecycle model allows for flexible intermediate consumption. Pension funds, especially collective Defined Contribution (D C) plans should take optimal consumption choice into consideration while making strategic decision on behalf of their participants. Earlier literatures document that proportional transaction costs reduce the rate of consumption, though some debate on whether the effect is weak or not.

We shall assume that pension funds can trade two assets continuously in an economy. The first asset is the money market account (the Bond) growing at a rate r as was in the case of Miao [23]. The second asset is a risky security (the stock). The pension fund takes these prices as given and chooses quantities without transaction costs. Further assumptions are that the securities pay no dividend and taxes on capital gains are zero. Following Merton [1], assume there is a single perishable consumption good as numeraire. The pension plan participants derive utility from inter-temporal consumption C of this good and the terminal wealth at time T. Pension plan participants are impatient. Their time preference can be summarized by a discount rate ρ. The consumption is made through the money market account. The participant has a CRRA utility function over consumption and terminal wealth. We ignore labor income in this context.

Throughout this paper, we are assume a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a filtration $\{\mathcal{F}_t\}$. Uncertainty in the models is generated by standard Brownian motion Z_t.

The two equations governing the dynamics of the money market account (bond) and stock are now given as;

$$dB_t = rB_t dt$$

or

$$B_t = B_o exp\{rt\} \qquad (1)$$

and

$$dS_t = \mu S_t dt + \sigma S_t dZ_t$$

or

$$S_t = S_o exp\left\{\sigma Z_t + \left(\mu - \frac{\sigma^2}{2}\right) t, \right\}, \forall\, t \in [0,1] \qquad (2)$$

The parameter B_0 is the initial investment on the money market account which determines the speed of a mean-reversion to the stationary level. σ is the acceleration coefficient which is the volatility (variance) of the process and is proportional to the level of the interest rate.

The admissible trading strategies are (D, I). The processes D and I are cumulative amount of sales and purchases of stock. The two processes satisfy $D(0) = I(0) = 0$, and both are non-decreasing, right continuous adapted. The evolution of the amount invested in the money market account and stock can be expressed as (Osu and Ihedioha, [24]):

$$\left.\begin{matrix} dB_t = rB_t dt - dI_t + dD_t \\ dS_t = \mu S_t dt + \sigma S_t dZ_t + dI_t - dD_t \end{matrix}\right\}. \qquad (3)$$

Pension funds all face a risky return trade-off of providing a safe pension at low cost. The decision making in a multiple member and multiple objective pension plans depends on the pension funds government, the financial position of the fund and risk attitudes and solvency positions, indexation quality and assets-liability risks are considered as primary objectives, Miao [17]. For tractability, quantitative derivation and insightful analytic solutions to optimal investment of a pension fund, we use CRRA UTILITY function of the final wealth, that is, $U(W) = \frac{W^{1-\gamma}}{1-\gamma}\ , for\ \ 0 < \gamma < 1. \gamma$ is the constant relative risk aversion parameter(that is the relative risk premium).

On behalf of the plan participants, the pension fund chooses optimal investment strategies D and I so as to maximize the final wealth at a deterministic time T.

Define the value function at time T as;

$$J(C, B, S, t; T) = \text{Max}_{(D,I)}\, E\left[\frac{(B_T+S_T)^{1-\gamma}}{1-\gamma}\right], \qquad (4)$$

where $W = B_T + S_T$ is the total investment from both the riskless and the risky assets.

Assumption1:

The parameter values satisfy:

$$0 < \frac{\mu - r}{\gamma\sigma^2} < 1. \qquad (5)$$

It guarantees that B and S would be chosen to be strictly positive.

Assumption 2*:* The participant makes intermediate consumption decision on the admissible consumption space $\mathbb{C}$, which satisfies $\int_0^t |c_s| ds < \infty, \forall t \in [0, T]$

Assumption 3*:* Consumption is made through the money market account.

The pension fund problem becomes:

$$J(C, B, S, t;\ T) \underset{(C_t,B_t,S_t:t>0)}{\text{Max}} E\left[\int_0^T e^{-\rho\tau}\frac{C_\tau{}^{1-\gamma}}{1-\gamma}d\tau + e^{-\rho\tau}\frac{(B_T+S_T)^{1-\gamma}}{1-\gamma}\right] \qquad (6)$$

Subject to;

$$dB_t = rB_t dt - C_t dt - dI_t + dD_t,$$

$$dS_t = \mu S_t dt + \sigma S_t dZ_t + dI_t - dD_t.$$

The constraints above are equivalent to:

$$dW_t = (rB_t + \mu S_t - C_t) + \sigma S_t dZ_t. \qquad (7)$$

The value function should also satisfy the terminal condition:

$$J(C, B, S, T;\ T) = \frac{(B_T + S_T)^{1-\gamma}}{1-\gamma}. \qquad (8)$$

The first term of the value function, J represents discounted utility from consumption flows, while the second term captures the idea that terminal wealth gives utility to the participant as well for he can finance his retirement consumption by using the benefit payment from time T upwards. To solve the optimal consumption and investment problem, the technique of stochastic dynamic optimization is used.

We start with the Bellman equation:

$$J(C, B, S, t;\ T) = \mathrm{Max}_{C,S}\left\{\frac{C^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho} E[J(C', B', S', t + \Delta t;\ T)]\right\}. \qquad (9)$$

The actual utility over the time interval of length Δt is $\frac{C^{1-\gamma}}{1-\gamma}\Delta t$, and the discounting over such time interval is expressed by $\frac{1}{1+\rho\Delta t}$. Therefore the Bellman equation becomes:

$$J(C, B, S, t;\ T) = \mathrm{Max}_{C,S}\left\{\frac{C^{1-\gamma}}{1-\gamma}\Delta t + \frac{1}{1+\rho\Delta t} E[J(C', B', S', t + \Delta t;\ T)]\right\}. \qquad (10)$$

Multiplying both LHS and RHS by a factor of $1 + \rho\Delta t$ and rearranging the terms, we get:

$$\rho J\Delta t = \underset{C,S}{\mathrm{Max}}\left\{\frac{C^{1-\gamma}}{1-\gamma}\Delta t(1 + \rho\Delta t) + E[\Delta J]\right\}. \qquad (11)$$

Dividing by Δt and let it go to 0, the Bellman equation becomes:

$$\rho J = \underset{S}{\mathrm{Max}}\left\{\frac{C^{1-\gamma}}{1-\gamma} + \frac{1}{dt} E[dJ]\right\}. \qquad (12)$$

Ito's lemma states: $dJ = \left(\frac{dJ}{dt} + (rB + \mu S - C)\frac{dJ}{dW} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 J}{dW^2}\right) dt + \sigma S \frac{dJ}{dW} dZ.$

Applying it to the Bellman equation, we get the corresponding HJB equation:

$$\frac{C^{1-\gamma}}{1-\gamma} + J_t + J_W(rB + \mu S - C) + \frac{1}{2} J_{WW} \sigma^2 S^2 - \rho J = 0 \quad (13)$$

We derive optimal consumption policy from the HJB equation. First order condition with respect to consumption on the HJB equation yields:

$$J_W = \frac{\partial}{\partial C} \frac{C^{1-\gamma}}{1-\gamma} = C^{-\gamma} \quad (14)$$

The optimal consumption is the given as:

$$C^* = (J_W)^{-\frac{1}{\gamma}} \quad (15)$$

Substituting the optimal consumption into the HJB equation yields:

$$\frac{c^{1-\gamma}}{1-\gamma} + J_t + J_W(rB + \mu S - C^*) + \frac{1}{2} J_{WW} \sigma^2 s^2 - \rho J = 0 \quad (16)$$

To eliminate B from the equation, use the condition $W = B + S$

$$\frac{C^{*1-\gamma}}{1-\gamma} + J_t + J_W(rW + (\mu - r)S - C^*) + \frac{1}{2} J_{WW} \sigma^2 S^2 - \rho J = 0. \quad (17)$$

We conjecture that the value function J must be linear to $\frac{W^{1-\gamma}}{1-\gamma}$, and takes the form; $J(C, B, S, t; T) = M(t; T). \frac{W^{1-\gamma}}{1-\gamma}$ for a horizon dependent function $M(t;T) > 0, \forall t \in [0, T]$.

Replacing C^* by $(J_w)^{-\frac{1}{\gamma}} = M^{-\frac{1}{\gamma}} W, ; J_t\ by\ M' \frac{W^{1-\gamma}}{1-\gamma}$;and J by $M' \frac{W^{1-\gamma}}{1-\gamma}$ in the HJB equation, it follows that:

$$\frac{M^{-\frac{1-\gamma}{\gamma}}}{1-\gamma} W^{1-\gamma} + M' \frac{W^{1-\gamma}}{1-\gamma} + MW^{-\gamma}\left(rW + (\mu - r)S - M^{-\frac{1}{\gamma}} W\right) - \frac{\gamma}{2} \mathrm{M}\mathrm{W}^{-\gamma-1} \sigma^2 \mathrm{S}^2 - \rho \mathrm{M} \frac{W^{1-\gamma}}{1-\gamma} = 0. \quad (18)$$

First order condition on s gives the optimal amount invested in stock:

$$S^* = \frac{\mu - r}{\gamma \sigma^2} W \quad (19)$$

We have the result that without transaction cost; optimal investment policy involves investing a constant fraction of wealth in the stock, independent of the investor's horizon. As long as $\mu > r$, the pension fund always holds the stock in its portfolio.

Allowing for intermediate consumption does not change optimal investment policy. The ratio of the amount invested in stock and money market account is:

$$\pi^* = \frac{S^*}{B^*} = \frac{\frac{\mu-r}{\gamma\sigma^2}W}{(1-\frac{\mu-r}{\gamma\sigma^2})W} = \frac{\mu-r}{\gamma\sigma^2-\mu+r}. \qquad (20)$$

Now replacing S with the optimal value $S^* = \frac{\mu-r}{\gamma\sigma^2}W$, in the HJB equation and rearrange, we find the ordinary differential equation of M in t as:

$$M^{-\frac{1-\gamma}{\gamma}}\frac{\gamma}{1-\gamma} + \frac{M'}{1-\gamma} + Mr + \frac{(\mu-r)^2}{2\gamma\sigma^2}M - \frac{\rho}{1-\gamma}M = 0 \ . \qquad (21)$$

Formalizing, we obtain:

$$\frac{dM}{dt} = -\gamma M^{-\frac{1-\gamma}{\gamma}} - \left((1-\gamma)r + \frac{(1-\gamma)(\mu-r)^2}{2\gamma\sigma^2} - \rho\right)M \qquad (22)$$

Lemma 1.

The value of the horizon dependent function $M(t,T)\ \forall\ t \in [0,T]$ is given as;

$$M(t,T) = A' e^{-\frac{1}{\gamma}\int_t^T \theta(\tau)d\tau} + \tau \ \text{,where} \quad \tau = T-t.$$

Proof:

Equation (22) is a Bernoulli equation of the form

$$\frac{dM}{dt} + \emptyset M = \gamma M^n, \qquad (23)$$

where,

$$\emptyset = \left((1-\gamma)r + \frac{(1-\gamma)(\mu-r)^2}{2\gamma\sigma^2} - \rho\right) \qquad (24a)$$

and

$$n = \frac{-(1-\gamma)}{\gamma}. \qquad (24b)$$

We divide through by M^n to get;

$$M^{-n}\frac{dM}{dt} + \emptyset M^{1-n} = \gamma \ . \qquad (25a)$$

We adopt a shorthand variable Z as follows;$Z = M^{1-n}$, so that,

$$\frac{dZ}{dt} = \frac{dZ}{dM}\frac{dM}{dt} = (1-n)M^{-n}\frac{dM}{dt} \quad . \qquad (25b)$$

The preceding equation (23) can now be written as;

$$\frac{1}{1-n}\frac{dZ}{dt} + \emptyset Z = \gamma \quad . \qquad (26)$$

We further multiply through by $(1-n)dt$ and rearrange (26) to get;

$$dZ + [(1-n)\emptyset Z - (1-n)\gamma]dt = 0. \qquad (27)$$

This is now a first-order linear differential equation in which the variable Z has taken the place of M. The solution of (26) is of form;

$$Z(t,T) = e^{-\int_t^T u d\tau}\left(A + \int_t^T W e^{\int_t^T u d\tau}\, d\tau\right). \qquad (28)$$

With $e^{-\int_t^T u d\tau} = e^{-(1-n)\emptyset t}$ and $W = (1-n)\gamma$, we have with (24b)

$$Z(t,T) = A e^{-\frac{1}{\gamma}\int_t^T \emptyset d\tau} + T - t. \qquad (29)$$

By (25a) we have;

$$M(t,T) = A' e^{-\frac{1}{\gamma}\int_t^T \emptyset d\tau} + T - t, \qquad (30)$$

to which we obtain, $M(T,T) \to 1$ (together with the terminal conditions) as $t \to T$.

The horizon dependent solution to the investment problem is;

$$J(C,B,S,t;\, T) = \frac{W^{1-\gamma}}{1-\gamma}\left(e^{-\frac{1}{\gamma}\int_t^T \emptyset d\tau} + T - t\right). \qquad (31)$$

Equation (27) is a linear differential equation with constant coefficient and constant term (Chiang and Wainwright,[25]) with solution;

$$Z(t) = \left(Z_0 - \frac{\gamma}{\emptyset}\right) e^{\frac{-\emptyset\tau}{\gamma}} + \frac{\gamma}{\emptyset}. \qquad (32a)$$

The substitution of $Z = M^{1-n}$ will then yield,

$$M^{\frac{1}{\gamma}} = \left(M_0^{\frac{1}{\gamma}} - \frac{\gamma}{\emptyset}\right) e^{\frac{-\emptyset\tau}{\gamma}} + \frac{\gamma}{\emptyset}, \qquad (32b)$$

where;

M_0 is the initial value of the horizon dependent function M. Notice that as $\tau \to \infty$ the exponential expression will approach zero. Consequently,

$$M^{\frac{1}{\gamma}} \to \frac{\gamma}{\theta}\ or M \to \left(\frac{\gamma}{\theta}\right)^{\gamma} \quad \text{as} \quad \tau \to \infty. \qquad (32c)$$

Therefore, the horizon dependent function will approach a constant as its equilibrium value. This equilibrium or steady-state value $\left(\frac{\gamma}{\theta}\right)^{\gamma}$, varies directly with γ, the propensity to invest and inversely with the growth rate function, θ.

Therefore, the horizon dependent function will approach a constant as its equilibrium value. This equilibrium or steady-state value $\left(\frac{\gamma}{\theta}\right)^{\gamma}$, varies directly with γ, the propensity to invest and inversely with the growth rate function, θ.

This is the maximized lifetime expected utility at time t under optimal investment policy, and at terminal date T, $J(C, B, S, t; T) = \frac{W^{1-\gamma}}{1-\gamma}$ as expected. Optimal consumption contains a horizon dependent fraction of wealth, which is dependent on the wealth at hand:

$$C_t^* = M(t, T)^{-\frac{1}{\gamma}} W_t \qquad (33)$$

3. Lifecycle investment with transaction costs

We consider lifecycle investment when transaction costs are present in this section. As shown by Constantinides ([20], [21]) and Taksar et al [26], among others had shown that an investment policy is simple in a sense that it is characterized by two reflecting barriers, the buy boundary P and the sell boundary O with $O < P$.The investor stops transacting as far as the portfolio ratio $\frac{B_t}{S_t}$ falls in the no-transaction region$[O, P]$, while he immediately transacts to the closest boundary when the ratio falls outside. In line with the proportional nature of transaction costs, the optimal trading size in continuous time model is always infinitely small so as to keep the portfolio ratio in the interval of no-transaction region.

To capture the idea that purchasing stock and bond both involves transaction costs, the proportional transaction cost rate $q,$ is defined as the amount of one asset the investor can buy by selling one unit amount of the other, with $0 \leq q < 1$,as in Dumas and Luciano [27], where they introduced the term "Conversion Ratio". This definition reflects the two way property of transaction costs.

Considering the proportional transaction costs described above, one may not feel too surprised to see that optimal investment policy is characterized by a no-transaction region, a buy region and a sell region. What is new message in the case of lifecycle model is that, if the introduction of transaction costs also has an impact on optimal consumption policy, then pension fund participants should take this into account when making consumption-savings decisions. In the simple lifecycle model of Constantinides [21], consumption policy is set to be a constant fraction of riskless asset. We release this assumption in the analysis that follows.

As for whether transaction costs affect consumption positively or negatively, we still have not reached any concrete conclusion. What we can say is that transaction costs have two opposite effects on consumption. The income effect depresses consumption since transaction costs deplete the capital gains and hence wealth at hand. On the other hand, the substitution effect shifts consumption to the earlier stage as current consumption becomes less costly than future consumption in terms of transaction costs.

We are now able to build a model to quantify the impact of transaction costs on optimal investment and consumption policy.

At each point in time, the three regions are identified depending on the portfolio ratio: the no-transaction region, the buy region, and the sell region (Davis and Norman,[3]; Liu and Loewenstein,[22]).

The pension fund's problem is:

$$J(C,B,S,t;T) = \text{Max}_{C_t,B_t,S_t:>0}\, E\left[\int_0^T e^{-\rho\tau}\frac{C_t 1-\gamma}{1-\gamma}d\tau + e^{-\rho T}\frac{(B_T+S_T)^{1-\gamma}}{1-\gamma}\right] \qquad (34)$$

Subject to:

$$dB = rB_t dt - C_t dt - dI_t + (1-k)dD_t$$

$$dS_t = \mu S_t dt + \sigma S_t dw_t + (1-k)dI_t - dD_t$$

The value function should also satisfy the terminal condition, that all the stock holding must be transformed to cash at time T:

$$J(C,B,S,t;T) = \frac{(B_T+S_T)^{1-\gamma}}{1-\gamma}. \qquad (35)$$

Assumption 4:

The value function $J\,(C,B,S,t;\,T)$ is once continuously differentiable in B and twice continuously differentiable in S.

The two boundary equations in the sell region and the buy region are as given below. In the no-transaction region, to obtain the HJB, apply Ito's lemma:

$$dJ = \left(\frac{dJ}{dt} + (rB-C)\frac{dJ}{dB} + \mu S\frac{dJ}{dS} + \frac{1}{2}\sigma^2 S^2\frac{d^2J}{dS^2}\right)dt + \sigma S\frac{dJ}{dS}dZ, \qquad (36)$$

to the Bellman equation (35). We have:

$$\frac{C^{1-\gamma}}{1-\gamma} + J_t + J_B(rB-C) + J_s\mu S + \frac{1}{2}J_{ss}\sigma^2 S^2 - \rho J = 0 \qquad O \le \frac{B}{S} \le P \qquad (37)$$

$$J_B = (1-q)J_S \qquad \frac{B}{S} > P \qquad (38)$$

$$(1-q)J_B = J_S \qquad \frac{B}{S} < O \qquad (39)$$

Substituting optimal consumption into HJB equation $C^* = (J_B)^{-\frac{1}{\gamma}}$ yields:

$$\frac{(J_B{}^{-\frac{1}{\gamma}})^{1-\gamma}}{1-\gamma} + J_t + \left(rB - J_B{}^{-\frac{1}{\gamma}}\right)J_B + \mu S J_S + \frac{1}{2}\sigma^2 S^2 J_{SS} - \rho J = 0 \;\; ; \;\; O \le \frac{b}{s} \le P \quad .(40)$$

The value function $J(C,B,S,t;\,T)$ is homogeneous of degree $1-\gamma$ for all positive numbers in (B,S), as shown in Fleming and Soner [28]. Define, $h = \frac{B}{S}$, for a new value function, $f\colon (-\infty,+\infty)\times[0,T] \to \mathcal{R}$, homogeneity gives:

$$J(C,B,S,t;\ T)\ = S^{1-\gamma}\, f(C,h,t;\ T)\ . \qquad (41)$$

The no-transaction region, buy region and sell region thus can be characterized by two horizon-dependent boundaries $O(t;\,T)$ and $P(t;\,T)$. We derive the new value function and its derivatives with respect to h and t by applying the chain rule, as follows:

$$J = S^{1-\gamma} f$$

$$J_t = S^{1-\gamma} f_t$$

$$J_B = S^{-\gamma} f_h$$

$$J_S = (1-\gamma)S^{-\gamma} f - BS^{-\gamma-1} f_h$$

$$J_{SS} = \gamma(1-\gamma)S^{-\gamma-1} f + 2\gamma BS^{-\gamma-2} f_h + B^2 S^{-\gamma-3} f_{hh}$$

Substituting the new value function and its derivatives with respect to h into the modified HJB equation and the two boundary equations on the buy region and the sell region, one obtains a system of partial differential equations (ODEs).

On the no-transaction region:

$$\frac{\gamma}{1-\gamma} f_h^{\,1-\frac{1}{\gamma}} + \frac{1}{2} f_{hh}\sigma^2 h^2 + f_h\big(\gamma\sigma^2 - (u-r)\big)h + f\left((1-\gamma)\left(\mu - \frac{\gamma\sigma^2}{2}\right) - \rho\right) + f_t = 0 \quad O(t;T) \le h \le P(t;T)\ . \qquad (42)$$

For $\frac{\gamma}{1-\gamma}\gamma f_h^{\frac{-1}{\gamma}} = (\gamma-1)\mu h,$ (42) becomes;

$$\frac{1}{2} f_{hh}\sigma^2 h^2 + f_h(\gamma\sigma^2 - (u-r) + (\gamma-1)\mu)h + f\left((1-\gamma)\left(\mu - \frac{\gamma\sigma^2}{2}\right) - \rho\right) + f_t = 0 \quad ,O(t;T) \le h \le P(t;T)\ . \qquad (43)$$

On the buy region:

$$\left(\frac{1}{1-q} + h\right) f_h(h,t;T) = (1-\gamma) f(h,t;T)\ ; \quad h > P(t,T). \qquad (44)$$

On the sell region:

$$(1-q+h) f_h(h,t;T) = (1-\gamma) f(h,t;T)\ ; \quad h < O(t,T) \qquad (45)$$

In addition, the following terminal condition must be satisfied

$$f(h,t;T) = \frac{(h_T+1)^{1-\gamma}}{1-\gamma}. \qquad (46)$$

3.1. **The solution of the Differential Equations**

This section provides the solutions to the differential equations (44), (45), and (46), and discusses the results.

Lifecycle investment: the case of constant rate of return

We start by stating the theorem;

THEOREM 1:

Let $f(h)$ be the value function of the pension funds with h as the prevailing money market account-stock ratio. Let $f(h)$ be twice continuously differentiable, the solution of the time-homogeneous value function equation (44) with;

$$f(0) = 0, \text{ and } f'(h) = 0 \tag{47}$$

is given by;

$$f(h) = c(\sigma h)^{\lambda_1} + \frac{h}{k - \frac{\xi(t)}{\sigma}}, \tag{48}$$

with

$$\frac{\hat{h}}{k - \frac{\xi(t)}{\sigma}} + cc\lambda_1 \left(\sigma \hat{h}\right)^{\lambda_1} = 0 \tag{49}$$

and

$$f(h) = \frac{1}{\xi(t) - \sigma k} \{\frac{\sigma^{1+\lambda_1} \dot{h}^{1-\lambda_1} h^{\lambda_1}}{\lambda_1} + \sigma h\}, \tag{50}$$

where $\hat{h}$is the expected optimal money market account-stock ratio for a period t, c is a constant and;

$$\lambda_1 = -\left[\frac{\xi}{\sigma} - \frac{1}{2}\right] + \left\{\left[\frac{\xi}{\sigma} - \frac{1}{2}\right]^2 + 2\,k\right\}^{\frac{1}{2}} \tag{51a}$$

$$\lambda_2 = -\left[\frac{\xi}{\sigma} - \frac{1}{2}\right] - \left\{\left[\frac{\xi}{\sigma} - \frac{1}{2}\right]^2 + 2\,k\right\}^{\frac{1}{2}}, \tag{51b}$$

are the positive and negative characteristic roots of (40) respectively.

Proof:

Let $\xi = (\gamma\sigma^2 - (\mu - r) + (\gamma - 1)\mu)$ and $k = \{ (\gamma - 1)\left(\mu - \frac{\gamma\sigma^2}{2}\right) + \rho\}$, then (40) reduces to the ode (with the conditions in Osu and Okoroafor, [29])

$$\frac{\sigma^2 h^2}{2} f_{hh} + \xi h f_h - kf = -h \qquad (52)$$

By the method of change of independent variables using Euler's substitution and solving by variation of parameters, the solutions obtained (Osu and Okoroafor [28]).An important relationship derived under the optimal condition is that the discount rate is proxy of the systematic volatility factor in the economy. So that the discounted rate gains from a unit investment at $\hat{h}$ equals the optimal unit $\bar{h}$ of ratio of money market account to stock for the expected optimal money market account to stock ratio $\hat{h}$.Therefore, by (47), we have;

$$f(\hat{h}) = c(\sigma\hat{h})^{\lambda_1} + \frac{\hat{h}}{k - \frac{\xi}{\sigma}} = \bar{h} \ . \qquad (53)$$

Solving for C in (49) and (53) and equating the results gives;

$$\hat{h} = \frac{\lambda_1[k - \frac{\xi}{\sigma}]}{\lambda_1 - 1} \ . \qquad (54)$$

Note: When the drift parameter ξ is large enough so that $\frac{\xi}{\sigma} > \frac{1}{2}$ then the right hand side of 48a is approximated by first order Taylor's expansion as

$\frac{k}{\frac{\xi}{\sigma} - \frac{1}{2}}$, thus, $\frac{\xi}{\sigma} - \frac{1}{2} = k$ and we obtain $\lambda_1 = 1$ and the optimal money market account-stock ratio of (54) is indeterminate.

Observe that for $\xi = \sigma k$, (50) becomes;

$$f(h) = \frac{1}{\xi(1-\sigma)}\left\{\frac{\sigma^{1+\lambda_1}\dot{h}^{1-\lambda_1} h^{\lambda_1}}{\lambda_1} + \sigma h\right\}, \qquad (55)$$

in which

$$\sigma_{1,2} = \pm\sqrt{\frac{2[(\mu-r)+\rho]}{\gamma(\gamma+1)}}, \qquad (56)$$

with the positive root given as;

$$\sigma_1 = \sqrt{\frac{2[(\mu-r)+\rho]}{\gamma(\gamma+1)}}. \qquad (57)$$

Equation (57) becomes zero for $\mu - r = -\rho$ and $\xi = (\gamma-1)\mu + \rho$ (the excess return of risky stock over the return from risk-less bond). Also, $(h) = f(0) = 0$.

This implies, if there is no risk, there is no investment and vice versa. Another implication in that the excess return of risky stock over the return from bonds equals the discounting rate. Again $\sigma_1 = 1$ if $\gamma = 1,$ and $\mu - r = 1 - \rho$ (that is, the difference in the rate of return of the risky and risk-less investment is the rate of discount (time preference) ρ less than unity). In this case , $f(h) = f(\infty) = \infty$.

On the sell and buy regions

We have;

$$\frac{1}{1-q} + h = 1 - q + h, \qquad (58)$$

which gives $q = 0$ or $q = 2$.

On the Buy Region:

$$f' = \frac{1-\gamma}{\frac{1}{1-q}+h} f. \qquad (59)$$

The variation of the value function with respect to the ratio of the investments the money market account-stock ratio, h, is $\frac{1-\gamma}{\frac{1}{1-q}+h} f$, for which we have

$$f(h) = k\left(\frac{1}{1-q} + h\right)^{1-\gamma}. \qquad (60)$$

Equation (59) becomes $f(h) = k(1+h)^{1-\gamma}$ for $q = 0$ and $f(h) = k(h-1)^{1-\gamma}$, for $q = 2$. For $k > 0$, $f(h)$ increases as $q \to 0$ and decreases as $q \to 2$. Thus if there is no transaction cost, growth rate of the value function is higher than when transaction cost.

On the Sell Region;

$$f' = \frac{(1-\gamma)}{1-q+h} \tag{61}$$

The variation of the value function with respect to the investment ratio h is $\frac{\gamma-1}{1-q+h}f$, which gives;

$$f(h) = k(1-q+h)^{1-\gamma}. \tag{62}$$

Generally, the value of the investment increases (decrease) in the sell region and decreases (increases) in the buy region for some values of $0 < q < 2$, but are equal when $q = 0$ or $q = 2$.

4. CONCLUSION

In this study the optional lifecycle investment policy for pension funds when regarding transaction costs is investigated. We have shown that in contrast to the infinite horizon case, the optional lifecycle investment policy represented by sell and buy boundaries is strongly horizon–dependent for an investor who attempts to maximize his utility on a finite horizon.

The case of no-transaction cost and proportional transaction costs are examined. According to the set of ordinary differential equations derived, for the no-transaction, region, buy and sell regions, it is confirmed that the optimal investment policy is horizon-dependent and closed form solution proffered.

In the no-transaction region, from (52), the value of the investment is zero which implies no risk no investment. Further, for $\sigma_1 = 1$ if $\gamma = 1,$ and $\mu - r = 1 - \rho$ (that is, the difference in the rate of return of the risky and risk-less investment is the rate of discount (time preference), ρ, less than unity).

A unit increment in variation on the buy region leads to a unit decrement variation on the sell region, vice versa.

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