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A SPECTRAL APPROACH TO PRICING OF FORWARD STARTING OPTIONS

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Abstract. We propose a new class of models for pricing forward starting options. We assume that the asset price is a nonlinear function of a CIR process, time changed by a composition of a Lévy subordinator and an absolutely continuous process. The new models introduce the nonlinearity in both drift and diffusion components of the underlying process and can capture jumps and stochastic volatility in a flexible way. By employing the spectral expansion technique, we are able to derive the analytical formulas for the forward starting option prices. We also implement a specific model numerically and test its sensitivity to some of the key parameters of the model.

Keywords: forward staring options; spectral expansion; Lévy subordinator; stochastic volatility; stochastic time change

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1. Introduction

The forward starting options are options that start at a specified future date with an expiration date set further in the future. Like standard options, forward starting options are paid in advance;

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however, the strike price is not determined until the specified future date before expiration. Forward starting options belong to the class of path-dependent exotic options and have many applications in finance. The forward starting options are the building blocks to the so-called cliquet or ratchet options, which can be seen as a series of consecutive forward starting options. Employee stock options, in a sense, can be treated as a type of forward starting options since their strike price is not fixed when the employee begins to work. The forward starting options are also used by insurance companies to manage the risk contained in guaranteed equity-linked life insurance products.

Even though the payoff function of the forward starting options appears simple, their pricing can be demanding. Rubinstein [27] provides a closed-form solution for forward starting options under the constant volatility assumption. Kruse and Nögel [15] adopt Heston's stochastic volatility model by integrating the option pricing formula with respect to the conditional density of the variance value at strike determination date. However, their quasi-analytical pricing formulas involve the numerical solution of a two-dimensional integration problem. Hong [12] demonstrates that the valuation of forward starting options can be further simplified to a single one-dimensional Fourier transform inversion as long as the characteristic function of the forward rate of return on the underlying asset is known. However, the accuracy and stability of Hong's approach depends on the optimization of a dampening factor that ensures square integrability. Nunes and Alcaria [24] propose an alternative and more robust pricing methodology that can be easily and efficiently implemented through a Gaussian quadrature. Their alternative pricing solution is derived under the general affine jump-diffusion (AJD) framework of Duffie et al. [10] and does not require any optimization routine. Zhang and Geng [35] provide an efficient method for pricing forward starting options under stochastic volatility model with double exponential jumps. The forward characteristic function of the log asset price is derived and thereby forward staring options are evaluated by Fourier-cosine technique. Other jump-diffusion or/and stochastic volatility models for forward starting options can be found in [3], [26], [34] and references therein.

In the previous studies on forward starting options, the underlying state variables are usually assumed to follow an AJD process where the drift, diffusion and jump intensity all have affine dependence on the state variables. Therefore, the analytical valuation of forward staring options in these models critically requires the knowledge of the exponential affine complex-valued transform considered by Duffie et al. [10]. In this paper, we relax the strong assumptions of the AJD models and propose a non-affine model with stochastic volatility and jumps for pricing forward starting options. First, we model the asset price as a power function of a square root process (CIR). The resulting model is called CIR-CEV model since the stochastic process of the asset price belongs to the nonlinear constant-elasticity-of-variance (CEV) model. The CIR-CEV model encompasses many important models such as CIR and 3/2 models of [4] as special cases. Second, to capture both jumps and stochastic volatility in the asset price, we take time change to the CIR-CEV model where the time change process is modeled by the composition of a Lévy subordinator and an absolutely continuous time change process. The resulting time changed CIR-CEV (TC-CIR-CEV) model allows for flexible specification of jump and stochastic volatility processes and can capture the nonlinearity in the asset price process.

The CIR-CEV model or TC-CIR-CEV model belongs to the nonlinear model and therefore, we can not apply the characteristic function approach to derive the analytical solution to the option pricing problem. Instead, we employ the spectral expansion or so-called eigenfunction expansion method that is particularly suited for pricing contingent claims written on symmetric Markov processes with time changes. For the underlying CIR process, we are able to obtain the explicit expressions for the eigenvalues and eigenfunctions by solving the associated Sturm-Liouville eigenvalue problem. We can then express the forward starting option prices in terms of spectral expansion where the expansion coefficients can be calculated in closed form. A remarkable result of spectral expansion method is that if the underlying diffusion process possesses the spectral expansion, then the time changed process has the expansion in the same eigenfunctions with eigenvalues replaced by the Laplace transform of the time change processes. We refer to [20] and [21] for the surveys on the spectral expansion method and [16]-[19], [22], [23] and [30]-[33] for its various applications.

The structure of the paper is as follows. In Section 2, we introduce the general framework for modeling the asset price as a power transformed time changed CIR process, where the time change process is the composition of a Lévy process and an absolutely continuous process.

In Section 3, we discuss the spectral expansion method for the proposed CIR-CEV and TC-CIR-CEV models. We also calculate some of the integrals that are key to the determination of the spectral expansion coefficients. In Section 4, we provide the analytical solutions to the forward starting option pricing problems with the tool of the spectral expansion. In Section 5, we analyze the effect of the parameters of the model on the option prices through specific numerical examples.

2. Model framework

Let $(\Omega, \mathscr{F}, Q)$ denote a probability space with an information filtration $(\mathscr{F}_t)$. We assume that under the risk-neutral measure Q, the state variable $X(t)$ is a CIR process, i.e.,

$$dX(t) = \kappa(\theta - X(t))dt + \sigma\sqrt{X(t)}dB(t) , \tag{2.1}$$

where κ, θ, σ are constants and $B(t)$ is a standard Brownian motion. To ensure X to be positive, we impose the Feller condition: $\alpha := \frac{2\theta\kappa}{\sigma^2} - 1 > 0$.

The asset price $S(t)$ is a power function of a time changed CIR process $Y(t)$, that is

$$S(t) = Y^{\beta}(t) , \tag{2.2}$$

where β is a constant satisfying $\beta > -(\alpha+1)$, $\beta \neq 0$ and

$$Y(t) = X(T(t)) , \tag{2.3}$$

where T is a time change process.

To introduce both jumps and stochastic volatility into the asset price, we model the time change process T by composing a Lévy subordinator and an absolutely continuous time change process as follows:

$$T(t) = T_1(T_2(t)) , \tag{2.4}$$

where T_1 is a Lévy subordinator to capture jumps and T_2 is an absolutely continuous time change process to produce stochastic volatility. We assume T_1 is independent of T_2 and both T_1 and T_2 are independent of $B(t)$ in (2.1).

The Lévy subordinator T_1 is a non-negative Lévy process with nondecreasing increments and the Laplace transform

$$\mathrm{E}[\exp(-\lambda \mathrm{T}_1(\mathrm{t}))] = \exp(-\mathrm{t}\phi(\lambda)) \, , \tag{2.5}$$

where ϕ is the Lévy exponent and given by the Lévy-Khintchine formula (see e.g., [28])

$$\phi(\lambda) = \gamma\lambda + \int_{(0,\infty)} (1-\exp(-\lambda s))\nu(ds) \, ,$$

where $\gamma \geq 0$ and the Lévy measure ν must satisfy

$$\int_{(0,\infty)} (s \wedge 1)\nu(ds) < \infty \, .$$

The absolutely continuous time change process T_2 is modeled as an integral of a positive process v, that is

$$T_2(t) = \int_0^t v(s)ds \, , \tag{2.6}$$

where v is also known as the activity rate process. In this paper, we require the Laplace transform of $T_2(t)$ can be expressed in closed form. We list below some of the processes that can act as activity rate processes:

- CIR process

$$dv(t) = \kappa_v(\theta_v - v(t))dt + \sigma_v\sqrt{v(t)}dB_v(t) \, , \tag{2.7}$$

  where $\kappa_v > 0$, $\theta_v > 0$, $\sigma_v > 0$ and $2\kappa_v\theta_v \geq \sigma_v^2$. $B_v(t)$ is a Brownian motion independent of $T_1(t)$ and $B(t)$ in (2.1).

- Squared Ornstein Uhlenbeck (OU) process

$$v(t) = w^2(t) \, , \tag{2.8}$$

  where

$$dw(t) = \kappa_w(\theta_w - w(t))dt + \sigma_w dB_w(t) \, , \tag{2.9}$$

  where $B_w(t)$ is a Brownian motion independent of $T_1(t)$ and $B(t)$ in (2.1).

- 3/2 process

$$dv(t) = \kappa_v(\theta_v - v(t))v(t)dt + \sigma_v v^{3/2}(t)dB_v(t) , \tag{2.10}$$

where $\kappa_v > 0$ and $\sigma_v > 0$. $B_v(t)$ is a Brownian motion independent of $T_1(t)$ and $B(t)$ in (2.1).

- Lévy-driven OU process

$$dv(t) = -\kappa_v v(t)dt + \sigma_v dL_v(t) , \tag{2.11}$$

where $\kappa_v > 0$ and $\sigma_v > 0$. $L_v(t)$ is a Lévy subordinator independent of $T_1(t)$ and $B(t)$ in (2.1).

For all of the above activity rate processes, not only the Laplace transform of T_2 but also the Laplace transform of the joint density of $v(t)$ and $T_2(t)$ can be calculated in closed form. For example, the following Lemma gives the Laplace transform of the joint density of $v(t)$ and $\int_0^t v(s)ds$ when $v(t)$ is a CIR process (see e.g., [14]):

Lemma 2.1. *For the CIR process defined in (2.7), the Laplace transform $\varphi(t,\alpha,\beta;v(0))$ of the joint density of $v(t)$ and $\int_0^t v(s)ds$ is given by*

$$\varphi(t,\alpha,\beta;v(0)) = \mathrm{E}\left[\exp\left(-\alpha \mathrm{v(t)} - \beta\int_0^{\mathrm{t}} \mathrm{v(s)ds}\right) \mid \mathrm{v(0)}\right] = \exp[\mathrm{A(t},\alpha,\beta) + \mathrm{B(t},\alpha,\beta)\mathrm{v(0)}] , \tag{2.12}$$

where

$$A(t,\alpha,\beta) = \frac{2\kappa_v\theta_v}{\sigma_v^2}\log\left(\frac{2\delta\exp((\delta+\kappa_v)t/2)}{\sigma_v^2\alpha(\exp(\delta t)-1)+\delta(\exp(\delta t)+1)+\kappa_v(\exp(\delta t)-1)}\right) , \tag{2.13}$$

and

$$B(t,\alpha,\beta) = -\frac{\alpha(\delta+\kappa_v+\exp(\delta t)(\delta-\kappa_v)+2\beta(\exp(\delta t)-1)}{\sigma_v^2\alpha(\exp(\delta t)-1)+\delta(\exp(\delta t)+1)+\kappa_v(\exp(\delta t)-1)} , \tag{2.14}$$

where $\delta = \sqrt{\kappa_v^2 + 2\sigma_v^2\beta}$.

Given the Laplace transform of T_2, we can immediately obtain the Laplace transform of the composite time change T through

$$\mathrm{E}[\exp(-\lambda \mathrm{T(t)})] = \mathrm{E}[\exp(-\lambda \mathrm{T_1(T_2(t))})] = \mathrm{E}\{\mathrm{E}[\exp(-\lambda \mathrm{T_1(T_2(t))}) \mid \mathrm{T_2(t)}]\} = \mathrm{E}[\exp(-\phi(\lambda)\mathrm{T_2(t)})] .$$

In the case $T(t) = t$, by Itô's Lemma, the dynamics of S is given by

$$dS(t) = \left[\left(\kappa\theta\beta + \frac{\sigma^2}{2}\beta(\beta-1)\right)S^{(\beta-1)/\beta}(t) - \kappa\beta S(t)\right]dt + \sigma\beta S^{(\beta-1/2)/\beta}(t)dB(t) \; . \tag{2.15}$$

It is clear that S belongs to the constant-elasticity-of-variance (CEV) process with nonlinear drift and diffusion. This CIR-CEV process has been employed by [5] and [7] for interest rates; [6] and [29] for VIX futures and options; [32] for electricity derivatives; [33] for discrete arithmetic Asian options. Many special cases of CIR-CEV models have been widely used in the finance literature. When $\beta = 1$, we obtain the CIR model which has been studied by [9] for interest rate and [8], [13] and [18] for commodities and electricity prices and derivatives. When $\beta = -1$, we obtain the 3/2 model for S which is a popular model for pricing interest rate and volatility derivatives; see e.g., [4] and [11]. When $\beta = 2$, we obtain the 3/4 model which has been used to model the crude oil futures and futures options in [1] and [2].

When $T(t) \neq t$, we obtain the TC-CIR-CEV model where the asset price S is constructed from the composition of two processes: CIR-CEV process $Y^\beta(t)$ and the time change process $T(t)$. This model can not only capture the nonlinearities in the asset price but also introduce jumps and stochastic volatility through stochastic time change. We refer to [32] and [33] for further applications of the TC-CIR-CEV model and [16]-[19], [22], [23], [30], [31] for the applications of other time changed processes in finance.

3. Spectral expansion method for the TC-CIR-CEV model

For the CIR process X in (2.1), its infinitesimal generator $\mathscr{L}$ is defined by

$$\mathscr{L}f(x) = \kappa(\theta - x)f'(x) + \frac{1}{2}\sigma^2 x f''(x) \; , \tag{3.1}$$

where f is transformation function. f' and f'' are first- and second-order derivatives of f, respectively.

The speed measure of X is given by

$$m(x) = \frac{2}{\sigma^2}x^\alpha \exp\left(-\frac{2\kappa x}{\sigma^2}\right) \; . \tag{3.2}$$

Define the inner product $(f,g) := \int_0^\infty f(x)g(x)m(x)dx$ and let $L^2((0,\infty),m)$ be the Hilbert space of functions on $(0,\infty)$ square-integrable with the speed density $m(x)$, that is, with $||f|| < \infty$, where $||f||^2 = (f,f)$.

Under some basic assumptions, the infinitesimal generator $\mathscr{L}$ with domain dom($\mathscr{L}$) is always self-adjoint on the Hilbert space $L^2((0,\infty),m)$. We can then appeal to the spectral theorem for self-adjoint operators in Hilbert space to obtain the spectral decomposition of the generator. For the process X, the spectrum of the negative of the infinitesimal generator $-\mathscr{L}$ is purely discrete. For any $f \in L^2((0,\infty),m)$, we can write down the spectral expansion or so-called eigenfunction expansion of the following expectation:

$$\mathrm{E}[\mathrm{f}(\mathrm{X}(\mathrm{t}))|\mathrm{X}(0)=\mathrm{x}] = \sum_{\mathrm{n}=0}^{\infty} \mathrm{f_n} \exp(-\lambda_{\mathrm{n}}\mathrm{t})\psi_{\mathrm{n}}(\mathrm{x}) \;, \tag{3.3}$$

where $f_n = (f,\psi_n)$, $\{\lambda_n\}_{n=0}^\infty$ are the eigenvalues of $-\mathscr{L}$ and $\{\psi_n\}_{n=0}^\infty$ are the corresponding eigenfunctions satisfying the following Sturm-Liouville equation

$$-\mathscr{L}\psi_n = \lambda_n\psi_n \;,$$

where the eigenfunctions $\{\psi_n\}_{n=0}^\infty$ form a complete orthonormal basis in the Hilbert space $L^2((0,\infty),m)$; that is, $(\psi_n,\psi_n) = 1$ and $(\psi_n,\psi_m) = 0$ for $n \neq m$.

For the CIR process X defined in (2.1), its eigenvalues and eigenfunctions can be summarized in the following result (see e.g., [21]):

Theorem 3.1. *For the CIR process in (2.1), the eigenvalues λ_n, $n = 0,1,\ldots$, are*

$$\lambda_n = \kappa n \;. \tag{3.4}$$

The eigenfunctions ψ_n, $n = 0,1,\ldots$, can be written as

$$\psi_n(x) = N_n L_n^{(\alpha)}\left(\frac{2\kappa x}{\sigma^2}\right) \;, \tag{3.5}$$

where $N_n^2 = \left(\frac{2\kappa}{\sigma^2}\right)^\alpha \frac{n!\kappa}{\Gamma(n+\alpha+1)}$. $L_n^{(\alpha)}(x)$ is the generalized Laguerre polynomials defined as

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1(-n;\alpha+1;x) \;,$$

where ${}_1F_1(a;b;x)$ *is the Kummer confluent hypergeometric function, given by*

$$ {}_1F_1(a;b;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!} \, , $$

where $(a)_0 = 1$, $(a)_n = a(a+1), \dots, (a+n-1)$, $n > 0$.

From (3.3), it is clear that time enters only through the exponentials $\exp(-\lambda_n t)$. Then under the TC-CIR-CEV model, for any $f \in L^2((0,\infty),m)$, we can employ the spectral expansion method to compute the following expectation (see e.g., [17]):

$$ \mathrm{E}[\mathrm{f}(\mathrm{Y}(\mathrm{t}))|\mathrm{Y}(0)=\mathrm{x}] = \mathrm{E}[\mathrm{f}(\mathrm{X}(\mathrm{T}(\mathrm{t})))|\mathrm{X}(0)=\mathrm{x}] = \sum_{\mathrm{n}=0}^{\infty} \mathrm{f_n} \mathrm{E}[\exp(-\lambda_{\mathrm{n}} \mathrm{T}(\mathrm{t}))] \psi_{\mathrm{n}}(\mathrm{x}) \, , \tag{3.6} $$

where Y is the process of X time changed by T. $\{\lambda_n\}$ and $\{\psi_n\}$ can be obtained from (3.4) and (3.5), respectively.

Thus, a key feature of the spectral expansion method is that the temporal and spatial variables are separated. The time variable t enters the expansion only through the exponential function $\exp(-\lambda_n t)$. The spectral expansion of time changed process Y has the same form as the original process X, but with $\exp(-\lambda_n t)$ replaced by $\mathrm{E}[\exp(-\lambda_{\mathrm{n}}\mathrm{T}(\mathrm{t}))]$. As long as the Laplace transform of the time change process T is known, the time changed model will be as tractable as the original model.

For the TC-CIR-CEV model, to calculate the forward staring option prices using the spectral expansion method, we need to compute the expansion coefficient f_n. We provide the formulas for the following integrals that will later be employed to calculate f_n.

- Define

$$ b_n(x,\beta) = \int_0^x y^{\beta} \psi_n(y) m(y) dy \, , $$

where $\beta > -(\alpha+1)$ and $x > 0$. The definitions of $\psi_n(y)$ and $m(y)$ can be found in (3.5) and (3.2), respectively.

- Define

$$ \tilde{b}_n(x,\beta) = \int_x^{\infty} y^{\beta} \psi_n(y) m(y) dy \, , $$

where $x > 0$.

- Define

$$a_{m,n}(k,\beta,\gamma)=\int_0^\infty y^{\beta\gamma}b_n(ky,(1-\gamma)\beta)\psi_n(y)\psi_m(y)m(y)dy\ ,$$

  where $\beta>-(\alpha+1)$ and $\gamma\in\{0,1\}$.

- Define

$$\tilde{a}_{m,n}(k,\beta,\gamma)=\int_0^\infty y^{\beta\gamma}\tilde{b}_n(ky,(1-\gamma)\beta)\psi_n(y)\psi_m(y)m(y)dy\ ,$$

  where $\beta>-(\alpha+1)$ and $\gamma\in\{0,1\}$.

We can prove the following results.

Lemma 3.2.

$$b_n(x,\beta)=\frac{2N_n(\alpha+1)_n}{\sigma^2 n!(\alpha+\beta+1)}x^{\alpha+\beta+1}{}_2F_2\left(\alpha+n+1,\alpha+\beta+1;\,\alpha+1,\alpha+\beta+2;\,-\frac{2\kappa}{\sigma^2}x\right)\ ,$$

where ${}_2F_2(a_1,a_2;\,b_1,b_2;\,x)$ is the generalized hypergeometric function given by

$${}_2F_2(a_1,a_2;\,b_1,b_2;\,x)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n}{(b_1)_n(b_2)_n}\frac{x^n}{n!}\ .$$

Proof. We have

$$b_n(x,\beta)=\frac{2N_n}{\sigma^2}\left(\frac{\sigma^2}{2\kappa}\right)^{\alpha+\beta+1}\int_0^{2\kappa x/\sigma^2}z^{\alpha+\beta}\exp(-z)L_n^{(\alpha)}(z)dz\ .$$

Using the following formula (see e.g., [25])

$$\int_0^x y^\gamma\exp(-ay)L_n^{(\alpha)}(ay)dy=\frac{(\alpha+1)_n}{n!(\gamma+1)}x^{\gamma+1}{}_2\mathrm{F}_2(\alpha+n+1,\gamma+1;\,\alpha+1,\gamma+2;\,-ax)\ ,$$

for $x>0$ and $\gamma>-1$, we immediately obtain the formula for b_n. □

Lemma 3.3.

$$\begin{aligned}\tilde{b}_n(x,\beta)=\frac{2N_n}{\sigma^2}\left(\frac{\sigma^2}{2\kappa}\right)^{\alpha+\beta+1}\Bigg[&-\left(\frac{2\kappa x}{\sigma^2}\right)^{\alpha+\beta+1}\frac{(1+\alpha)_n}{n!(\alpha+\beta+1)}\\&\times{}_2F_2\left(\alpha+\beta+1,\alpha+n+1;\,\alpha+\beta+2,\alpha+1;\,-\frac{2\kappa}{\sigma^2}x\right)\\&+\frac{(-\beta)_n\Gamma(\alpha+\beta+1)}{n!}{}_2F_2\left(0,-\beta+n;\,-\alpha-\beta,-\beta;\,-\frac{2\kappa}{\sigma^2}x\right)\Bigg]\ .\end{aligned}$$

Proof. We have

$$\tilde{b}_n(x,\beta) = \frac{2N_n}{\sigma^2}\left(\frac{\sigma^2}{2\kappa}\right)^{\alpha+\beta+1}\int_{2\kappa x/\sigma^2}^{\infty} z^{\alpha+\beta}\exp(-z)L_n^{(\alpha)}(z)dz\,.$$

Using the following formula (see e.g., [25])

$$\begin{aligned}\int_x^{\infty} y^{\gamma}\exp(-ay)L_n^{(\alpha)}(ay)dy = &-\frac{(\alpha+1)_n}{n!(\gamma+1)}x^{\gamma+1}{}_2\mathrm{F}_2(\alpha+n+1,\gamma+1;\,\alpha+1,\gamma+2;\,-ax)\\ &+\frac{a^{-\gamma-1}}{n!}(\alpha-\gamma)_n\Gamma(\gamma+1){}_2\mathrm{F}_2(0,\alpha-\gamma+n;\,-\gamma,\alpha-\gamma;\,-ax)\,,\end{aligned}$$

for $x>0$ and $a>0$, we immediately obtain the formula for $\tilde{b}_n$. □

Lemma 3.4.

$$\begin{aligned}a_{m,n}(k,\beta,\gamma) = &\frac{4N_n^2N_m(\alpha+1)_n}{\sigma^4 n!(\alpha+(1-\gamma)\beta+1)}\left(\frac{\sigma^2}{2\kappa}\right)^{2\alpha+\beta+2}k^{\alpha+(1-\gamma)\beta+1}(-1)^{m+n}\\ &\times\sum_{s=0}^{m+n}\frac{C_s(m,n,\alpha-m+n,\alpha+m-n)}{s!}\Gamma(2\alpha+\beta+2+s)\\ &\times{}_3F_2\left(2\alpha+\beta+2+s,\alpha+n+1,\alpha+(1-\gamma)\beta+1;\,\alpha+1,\alpha+(1-\gamma)\beta+2;\,-k\right)\,,\end{aligned}$$

where C_s is the function defined by

$$C_s(m,n,\alpha,\beta) = (-1)^{m+n+s}\sum_{d=0}^{s}\binom{s}{d}\binom{m+\alpha}{n-s+d}\binom{n+\beta}{m-d}\,, \tag{3.7}$$

and ${}_3F_2(a_1,a_2,a_3;\,b_1,b_2;\,x)$ is the generalized hypergeometric function given by

$${}_3F_2(a_1,a_2,a_3;\,b_1,b_2;\,x) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n}\frac{x^n}{n!}\,.$$

Proof. We have

$$\begin{aligned}a_{m,n}(k,\beta,\gamma) = &\frac{4N_n^2N_m(\alpha+1)_n}{\sigma^4 n!(\alpha+(1-\gamma)\beta+1)}\left(\frac{\sigma^2}{2\kappa}\right)^{2\alpha+\beta+2}k^{\alpha+(1-\gamma)\beta+1}\\ &\times\int_0^{\infty} z^{2\alpha+\beta+1}{}_2\mathrm{F}_2\left(\alpha+n+1,\alpha+(1-\gamma)\beta+1;\,\alpha+1,\alpha+(1-\gamma)\beta+2;\,-kz\right)\\ &\times L_m^{(\alpha)}(z)L_n^{(\alpha)}(z)\exp(-z)dz\,.\end{aligned}$$

Using the following formula

$$L_m^{(\alpha)}(x)L_n^{(\beta)}(x) = (-1)^{m+n}\sum_{s=0}^{m+n}C_s(m,n,\beta-m+n,\alpha+m-n)\frac{x^s}{s!}\,,$$

where C_s is defined in (3.7) and the formula (see e.g., [25])

$$\int_0^\infty x^{\alpha-1}\exp(-cx)_2\mathrm{F}_2\left(a_1,a_2;b_1,b_2;-x\right)dx = c^{-\alpha}\Gamma(\alpha)_3\mathrm{F}_2\left(\alpha,a_1,a_2;b_1,b_2;-\frac{1}{c}\right),$$

for $\alpha > 0$ and $c > 0$, we obtain the formula for $a_{m,n}$. $\square$

Lemma 3.5.

$$\begin{aligned}\tilde{a}_{m,n}(k,\beta,\gamma) = -a_{m,n}(k,\beta,\gamma) + &\frac{4N_n^2 N_m(-(1-\gamma)\beta)_n\Gamma(\alpha+(1-\gamma)\beta+1)}{\sigma^4 n!}\left(\frac{\sigma^2}{2\kappa}\right)^{2\alpha+\beta+2}\\ &\times(-1)^{m+n}\sum_{s=0}^{m+n}\frac{C_s(m,n,\alpha-m+n,\alpha+m-n)}{s!}\Gamma(\beta\gamma+\alpha+s+1)\\ &\times{}_3F_2\left(\beta\gamma+\alpha+s+1,0,-(1-\gamma)\beta+n;-\alpha-(1-\gamma)\beta,-(1-\gamma)\beta;-k\right).\end{aligned}$$

The proof of Lemma 3.5 is similar to Lemma 3.4, so we omit it.

4. Valuation of forward starting options

A forward staring option is a contract in which the holder receives at the strike determination time t_1 an option with expiry date $t_2 > t_1$ and exercise price $kS(t_1)$ for some constant $k > 0$. The terminal payoff of a forward starting put option is thus given by the following:

$$(kS(t_1) - S(t_2))^+ .$$

The payoff function of the forward starting option looks simple, but the valuation of the option is demanding since typically we need the models that can capture stochastic volatility and/or jumps. Fortunately, for the TC-CIR-CEV model developed in Section 2, we are able to derive the analytical formula for forward staring option prices by employing spectral expansion method. For illustration purpose, we will focus on the example where the activity rate process v for the absolutely continuous time change process T_2 is CIR. We can write down the formula for the time $t_0 = 0$ value of a forward staring put option $P(t_1,t_2;k,s_0,v_0))$ with exercise price $kS(t_1)$ and expiry date t_2 ($t_0 < t_1 < t_2$), conditioning on $S(t_0) = s_0$ and $v(t_0) = v_0$, as follows:

Theorem 4.1. *Assume stochastic process for $S(t)$ is specified in (2.1)-(2.3) and time change process $T(t) = T_1(T_2(t))$, where T_1 is a Lévy subordinator with Lévy exponent ϕ and T_2 is an* *integrated CIR process defined in (2.6) and (2.7), then price of the forward starting put option* $P(t_1,t_2;k,s_0,v_0)$ *is*

1. If $\beta > 0$*, then*

$$
\begin{aligned}
&P(t_1,t_2;k,s_0,v_0)\\
&= \exp(-rt_2)\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(ka_{m,n}(k^{1/\beta},\beta,1) - a_{m,n}(k^{1/\beta},\beta,0)\right)\exp\Big[A(t_2-t_1,0,\phi(\lambda_n))\right.\\
&\quad\left. + A(t_1,-B(t_2-t_1,0,\phi(\lambda_n)),\phi(\lambda_m)) + B(t_1,-B(t_2-t_1,0,\phi(\lambda_n)),\phi(\lambda_m))v_0\Big]\psi_m\left(s_0^{1/\beta}\right)\right\},
\end{aligned}
$$

where the function $a_{m,n}$ *can be obtained from Lemma 3.4. A and B can be found in Lemma 2.1.*

2. If $-(\alpha+1) < \beta < 0$*, then*

$$
\begin{aligned}
&P(t_1,t_2;k,s_0,v_0)\\
&= \exp(-rt_2)\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(k\tilde{a}_{m,n}(k^{1/\beta},\beta,1) - \tilde{a}_{m,n}(k^{1/\beta},\beta,0)\right)\exp\Big[A(t_2-t_1,0,\phi(\lambda_n))\right.\\
&\quad\left. + A(t_1,-B(t_2-t_1,0,\phi(\lambda_n)),\phi(\lambda_m)) + B(t_1,-B(t_2-t_1,0,\phi(\lambda_n)),\phi(\lambda_m))v_0\Big]\psi_m\left(s_0^{1/\beta}\right)\right\},
\end{aligned}
$$

where the function $\tilde{a}_{m,n}$ *can be obtained from Lemma 3.5.*

Proof. To prove 1, using iterated conditional expectation, we have

$$
\begin{aligned}
&P(t_1,t_2;k,s_0,v_0)\\
&= \exp(-rt_2)\mathrm{E}[(\mathrm{kS}(\mathrm{t}_1) - \mathrm{S}(\mathrm{t}_2))^+]\\
&= \exp(-rt_2)\mathrm{E}\{\mathrm{E}[(\mathrm{kS}(\mathrm{t}_1) - \mathrm{S}(\mathrm{t}_2))^+|\mathscr{F}_{\mathrm{t}_1}]\}\\
&= \exp(-rt_2)\mathrm{E}\{\mathrm{kS}(\mathrm{t}_1)\mathrm{E}[1_{\{\mathrm{S}(\mathrm{t}_2)<\mathrm{kS}(\mathrm{t}_1)\}}|\mathscr{F}_{\mathrm{t}_1}]\\
&\qquad - \mathrm{E}[\mathrm{S}(\mathrm{t}_2)1_{\{\mathrm{S}(\mathrm{t}_2)<\mathrm{kS}(\mathrm{t}_1)\}}|\mathscr{F}_{\mathrm{t}_1}]\}\ .
\end{aligned}
\tag{4.1}
$$

The two expectations in the last line can be calculated using spectral expansion. Using Lemma 2.1, we have

$$
\begin{aligned}
&\mathrm{E}[1_{\{\mathrm{S(t_2)<kS(t_1)}\}}|\mathscr{F}_{\mathrm{t_1}}] \\
&= \mathrm{E}[1_{\{\mathrm{Y(t_2)<k^{1/\beta}Y(t_1)}\}}|\mathscr{F}_{\mathrm{t_1}}] \\
&= \sum_{n=0}^{\infty} b_n(k^{1/\beta}Y(t_1),0)\mathrm{E}[\exp(-\lambda_{\mathrm{n}}\mathrm{T(t_2-t_1)})]\psi_{\mathrm{n}}(\mathrm{X(T(t_1))}) \\
&= \sum_{n=0}^{\infty} b_n(k^{1/\beta}Y(t_1),0)\exp[A(t_2-t_1,0,\phi(\lambda_n))+B(t_2-t_1,0,\phi(\lambda_n))v(t_1)]\psi_n(X(T(t_1)))\ , \quad (4.2)
\end{aligned}
$$

where we used

$$b_n(k^{1/\beta}Y(t_1),0) = \int_0^{k^{1/\beta}Y(t_1)} \psi_n(y)m(y)dy\ .$$

Similarly, we have

$$
\begin{aligned}
&\mathrm{E}[\mathrm{S(t_2)}1_{\{\mathrm{S(t_2)<kS(t_1)}\}}|\mathscr{F}_{\mathrm{t_1}}] \\
&= \mathrm{E}[\mathrm{Y}^{\beta}\mathrm{(t_2)}1_{\{\mathrm{Y(t_2)<k^{1/\beta}Y(t_1)}\}}|\mathscr{F}_{\mathrm{t_1}}]
\end{aligned}
$$

(4.3)

$$= \sum_{n=0}^{\infty} b_n(k^{1/\beta}Y(t_1),\beta)\exp[A(t_2-t_1,0,\phi(\lambda_n))+B(t_2-t_1,0,\phi(\lambda_n))v(t_1)]\psi_n(X(T(t_1)))\ ,$$

where we used

$$b_n(k^{1/\beta}Y(t_1),\beta) = \int_0^{k^{1/\beta}Y(t_1)} y^{\beta}\psi_n(y)m(y)dy\ .$$

Plugging (4.2) and (4.3) into (4.1) and using spectral expansion and Lemma 2.1 again, we obtain

$$
\begin{aligned}
&P(t_1,t_2;k,s_0,v_0) \\
&= \exp(-rt_2)\mathrm{E}\Bigg\{\mathrm{kY}^{\beta}\mathrm{(t_1)}\sum_{\mathrm{n=0}}^{\infty}\mathrm{b_n(k^{1/\beta}Y(t_1),0)}\exp\Big[\mathrm{A(t_2-t_1,0,}\phi(\lambda_{\mathrm{n}}))\\
&\quad +B(t_2-t_1,0,\phi(\lambda_n))v(t_1)\Big]\psi_n(X(T(t_1))) - \sum_{n=0}^{\infty} b_n(k^{1/\beta}Y(t_1),\beta)\\
&\quad \times\exp\Big[A(t_2-t_1,0,\phi(\lambda_n))+B(t_2-t_1,0,\phi(\lambda_n))v(t_1)\Big]\psi_n(X(T(t_1)))\Bigg\}
\end{aligned}
$$

$$= \exp(-rt_2)\left\{ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left(ka_{m,n}(k^{1/\beta},\beta,1) - a_{m,n}(k^{1/\beta},\beta,0)\right) \exp\left[A(t_2-t_1,0,\phi(\lambda_n))\right.\right.$$

$$\left.\left. + A(t_1,-B(t_2-t_1,0,\phi(\lambda_n)),\phi(\lambda_m)) + B(t_1,-B(t_2-t_1,0,\phi(\lambda_n)),\phi(\lambda_m))v_0\right]\psi_m\left(s_0^{1/\beta}\right)\right\}.$$

where we used

$$a_{m,n}(k^{1\beta},\beta,1) = \int_0^\infty y^\beta b_n(k^{1\beta}y,0)\psi_n(y)\psi_m(y)m(y)dy\ ,$$

and

$$a_{m,n}(k^{1\beta},\beta,0) = \int_0^\infty b_n(k^{1\beta}y,\beta)\psi_n(y)\psi_m(y)m(y)dy\ .$$

The proof to 2 is similar to 1 and we omit it. □

5. Numerical Analysis

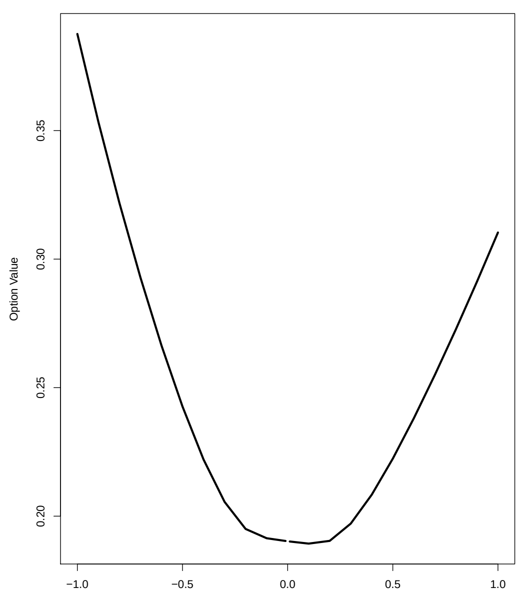

FIGURE 1. Option prices for different β. The parameters are $\kappa = 3,\ \theta = 1,\ \sigma = 1,\ \kappa_v = 4,\ \theta_v = 1,\ \sigma_v = 1,\ \vartheta = 1,\ \omega = 0.02,\ r = 0.05,\ s_0 = 1,\ v_0 = 1,\ k = 1.2,\ t_1 = 0.5$ and $t_2 = 1$.

In this section, we numerically study forward starting option pricing based on a specific TC-CIR-CEV model. We will focus on the put options. We assume the time change process

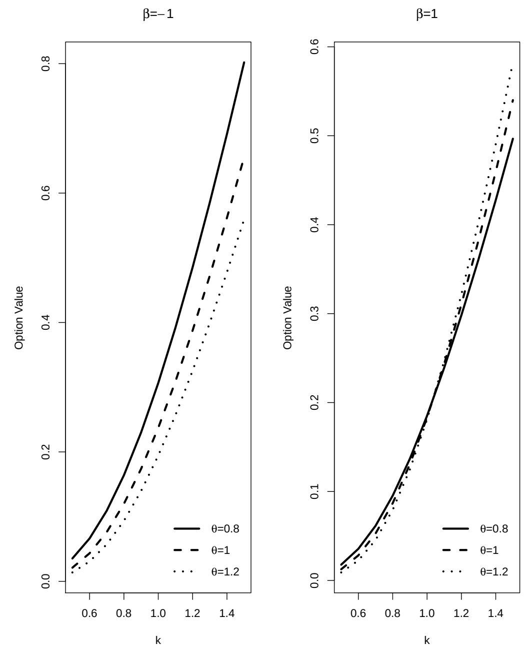

FIGURE 2. Option prices for different θ. The parameters are $\kappa = 3$, $\sigma = 1$, $\kappa_v = 4$, $\theta_v = 1$, $\sigma_v = 1$, $\vartheta = 1$, $\omega = 0.02$, $r = 0.05$, $s_0 = 1$, $v_0 = 1$, $t_1 = 0.5$ and $t_2 = 1$.

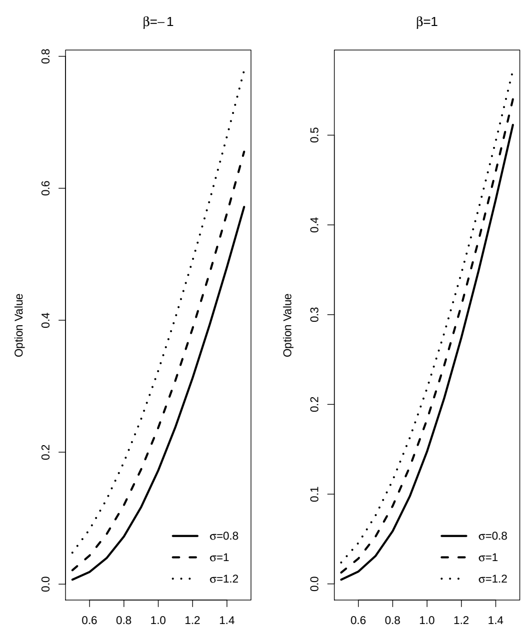

FIGURE 3. Option prices for different σ. The parameters are $\kappa = 3$, $\theta = 1$, $\kappa_v = 4$, $\theta_v = 1$, $\sigma_v = 1$, $\vartheta = 1$, $\omega = 0.02$, $r = 0.05$, $s_0 = 1$, $v_0 = 1$, $t_1 = 0.5$ and $t_2 = 1$.

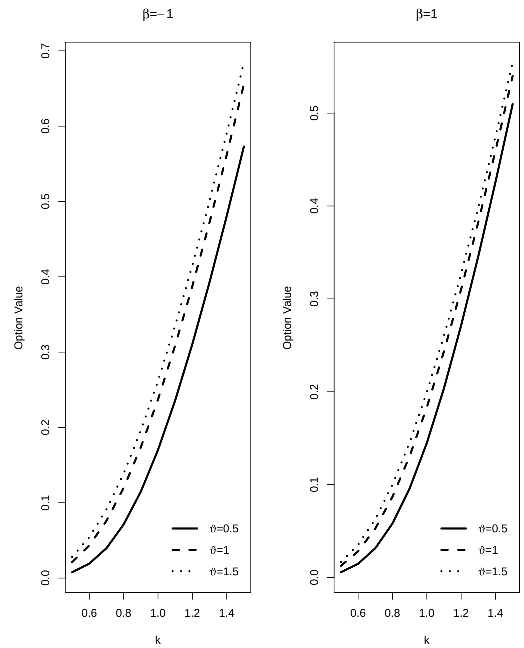

FIGURE 4. Option prices for different ϑ. The parameters are $\kappa = 3$, $\theta = 1$, $\sigma = 1$, $\kappa_v = 4$, $\theta_v = 1$, $\sigma_v = 1$, $\omega = 0.02$, $r = 0.05$, $s_0 = 1$, $v_0 = 1$, $t_1 = 0.5$ and $t_2 = 1$.

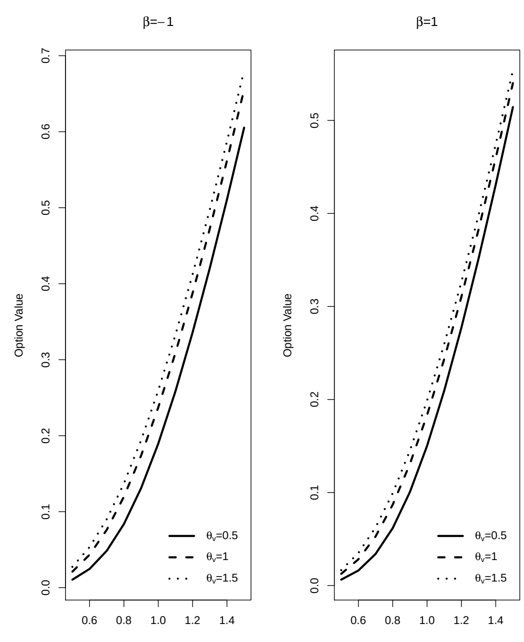

FIGURE 5. Option prices for different θ_v. The parameters are $\kappa = 3$, $\theta = 1$, $\sigma = 1$, $\kappa_v = 4$, $\sigma_v = 1$, $\vartheta = 1$, $\omega = 0.02$, $r = 0.05$, $s_0 = 1$, $v_0 = 1$, $t_1 = 0.5$ and $t_2 = 1$.

$T(t) = T_1(T_2(t))$, where T_2 is an integrated CIR process and T_1 is a Gamma subordinator with the Lévy exponent

$$\phi(\lambda) = \frac{\vartheta^2}{\omega} \log\left(1 + \frac{\omega\lambda}{\vartheta}\right),$$

where $\vartheta = \mathrm{E(T(1))}$ and $\omega = \mathrm{Var(T(1))}$.

We employ the spectral expansion method to calculate the option prices. In practice, we need to truncate the eigenfunction expansion after a finite number of terms. We will following [16] by truncating the infinite series when a given error tolerance level is reached. In the base case of the numerical analysis, we select the following parameter values: $\kappa = 3$, $\theta = 1$, $\sigma = 1$, $\kappa_v = 4$, $\theta_v = 1$, $\sigma_v = 1$, $\vartheta = 1$, and $\omega = 0.02$. We find the convergence of the expansion is really fast under these parameters.

In Fig.1, we plot the option prices for different values of power transformation parameter β. It is clear the option prices are sensitive to β. Different β implies different levels of mean and volatility for the asset prices. It seems that the mean of the underlying asset price increases with β when $\beta > 0$ and decreases with β when $\beta < 0$. As a result, the option values decline with respect to β when $\beta < 0$, but increase when $\beta > 0$.

We also evaluate the effect of other parameter value changes on the forward starting option prices. We consider the sensitivity of option prices with respect to four parameters: the long term mean parameter θ for the process X, the volatility term σ for the process X, the mean parameter ϑ for the time change process $T_1(1)$ and the mean parameter θ_v for the activity rate process v of T_2. We plot the forward starting option prices against different values of excise price constant k for both $\beta = -1$ and $\beta = 1$.

When β is negative (positive), the mean of asset price S is a decreasing (increasing) function of mean of process X. The prices of forward staring put options, however, will depend on the mean of the difference between $kS(t_1)$ and $S(t_2)$. Therefore, we expect the relationship between θ and the option prices will be dependent on the values of k and may not be monotone. From Fig. 2, we find for the given parameters, the option prices decrease with θ in the case $\beta = -1$. When $\beta = 1$, the prices decrease with θ when k is small but increase when k becomes larger.

From Figs. 3-5, it is clear that the option prices increase with σ, ϑ and θ_v in both $\beta = -1$ and $\beta = 1$ cases. These are as expected since when σ, ϑ and θ_v increase, the variability in the asset prices will be greater no matter whether β is negative or positive. When the variability increases, the option prices will also increase.

6. Conclusion

In the classical models for the valuation of forward starting options, the underlying state variables are usually assumed to follow the AJD process where the drift, diffusion and jump intensity all have affine dependence on the state variables. In this paper, we propose a new nonlinear CEV model for the purpose of pricing forward starting options. The asset price is modeled by a power transformed time changed CIR process, where the time change process is a composition of a Lévy subordinator and an absolutely continuous process. The resulting model introduces nonlinearity in both drift and diffusion parts of the underlying process and allows for flexible forms of jump and stochastic volatility processes. We employ spectral expansion method to obtain the analytical formulas for the prices of forward starting options. We also numerically implement our model and test its sensitivity to some of the key parameters of the underlying processes.

Conflict of Interests

The authors declare that there is no conflict of interests.

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