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A STOCHASTIC ALGORITHM FOR THE VALUATION OF FINANCIAL DERIVATIVES USING THE HYPERBOLIC DISTRIBUTIONAL VARIATES

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Abstract: It is a well-known fact that the difference between the continuous compounding rate of returns of financial derivatives X_t and its geometric rate of returns Y_t is negligible if X_t is typically of $O(10^{-2})$. The aim of this paper is to find the value of this difference when X_t is not negligible. We first establish that X_t and hence Y_t are distributed according to the generalized hyperbolic distribution (GH_d) to accommodate linear transformation property. We then apply a stochastic algorithm to trace the non-zero value of X_t and hence the value of Y_t and their difference. An illustrative example is given in concrete setting.

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1. Introduction

In valuation and options pricing theories for derivative securities as well as in other question in finance, the distributional form of the returns on the underlying assets plays a key role. For many years both financial economists and statisticians have been concerned with description of stock market returns. The form of the distribution of stock returns is a crucial assumption for mean-variance portfolio theory, theoretical models of capital asset prices, and the prices of contingent claims. For example, understanding the behaviour of the variance is essential to option pricing models.

It is widely known that the assumption according to which the financial asset returns are normally distributed is not supported by empirical evidence. Cont (2001), concludes that the precise form of the tail of financial returns distribution is difficult to determine, and in order for a parametric distributional model to reproduce the properties of the empirical distribution it must have at least four parameters: a location parameter, a scale parameter, a parameter describing the decay of the tails and an asymmetry parameter, therefore it is important to develop theoretical models based on other distribution classes, explaining asymmetry and heavy-tail phenomena. In this sense, stable distributions and normal mixture distributions have been used with considerable success.

Let $(S_t)_{t=0}$ denote the price process of a security, in particular of a stock. The rate of daily arithmetic returns are defined by (the continuous compounding)

$$X_t = \frac{(S_t - S_{t-1})}{S_{t-1}}, \tag{1}$$

and the yearly arithmetic returns are defined by

$$X_t = \frac{(S_T - S_0)}{S_0},$$

where S_0 and S_T are the prices of the security at the first and last trading day of the year respectively, we have that x_t may be

$$x_t = \frac{S_t}{S_0} - 1 = \frac{S_t(S_{t-1})}{S_{t-1}(S_{t-2})} \dots \frac{S_1}{S_0} - 1 = \prod_{t=1}^{T} \frac{S_T}{S_{t-1}} - 1, \tag{2}$$

That this, we have describe the yearly arithmetic return as a function, or a sum of daily arithmetic returns.

The daily geometric returns are defined by

$$y_t = \log(S_t) - \log(S_{t-1}),$$

while the yearly geometric returns are given by

$$Y_t = \log(S_t) - \log(Y_0). \tag{3}$$

we can as well write Y_t as

$$Y_t = \log\left(\prod_{t-1}^{T} \frac{S_T}{S_{t-1}}\right)$$
$$= \sum_{t-1}^{T} \log\left(\frac{S_T}{S_{t-1}}\right) = \sum_{t-1}^{T} Y_t \; . \tag{4}$$

This means that yearly geometric returns are equal to the sum of daily geometric returns (Aas and Dimakos, 2004)

Notice that from (4)

$$Y_t + \cdots + Y_{t+n-1} = \log S_{t+n-1} - \log S_{t-1} \tag{5}$$

which does not hold for X_t. The underlying price process is a continuous-time process from which discrete time series are drawn at equidistant time point but for continuous time processes returns with continuous compounding are the natural choice. The fact that the underlying process is a continuous time process leads to the use of t both as a continuous and as a discrete parameter. Eberlain and Keller(1995) had identified distributional form of compound returns as the generalized Hyperbolic distribution GH_d, since it has been observed that actual returns distributions appear fat-tailed (compared to Normal) and skewed.

The relationship between (2) and (4) is

$$Y_t = \log(1 + X_t), \tag{6}$$

which can be decomposed into a Taylor series as

$$Y_t = X_t + \frac{1}{2}X_t^2 + \frac{1}{3}X_t^3 +, \ldots, \tag{7a}$$

or

$$Y_t - X_t = \frac{1}{2}X_t^2 + \frac{1}{3}X_t^3 +, \ldots, \tag{7b}$$

which simplifies to zero if X_t is small. That is the difference between X_t and Y_t is negligible when X_t is typically of the order 10^{-2}. If X_t is small, this means in practice that volatility of a price series is small , and the time resolution is high and geometric and arithmetic returns are quite similar, but when volatility increases and the time resolution decreases, the differences grows larger. The question here; what is the numerical differences between X_t and Y_t if the time resolution decreases and the differences grow large.

This question is what this study seeks to address. For portfolio management, the practical utility of non-normal distribution like GH requires two things;

I. There must be a fact algorithm for calibrating the parameters to data and
II. The distribution family must be closed under linear combinations (transformations).

Under this linear transformation, we propose a stochastic algorithm to address the question above.

Our proposed method has been successfully used in solving linear system and it is capable of overcoming the sparcity difficulty and also reaching the solution at one iteration. Thus we are going to apply a stochastic approximation method introduced by Okoroafor and Ekere (1999) and have been extremely studied by other authors (see for example Okorafor and Osu, 2004,2005).

2. Option's Pricing with the Generalized Hyperbolic Distribution

We consider a market model with a riskless asset M on n risky assets $S^i, i = i, \ldots, n$. We make the usual assumptions that there exists a probability space $(\Omega, F, \mathbb{P})$, a complete and right continuous filtrations $(\mathcal{F}_t)_{t\in[0,1]}$, where $\mathcal{F}_t$ represents the information available up to time t and that S^i are stochastic processes adapted to $(\mathcal{F}_t)_t$. Moreover, we assume that there exist a probability measure Q equivalent to P which is called forward neutral probability (El Karoui et al, 1995), such that the value of the riskless asset m remains constantly equal to i through time and the dynamics of prices of assets S^i under Q are the following

$$ds_t^i = s_t^i \langle \sigma_t^i dW_t \rangle,$$

where $(W_t)_t$ is a d-dimensional Q-Brownian motion adapted to $(\mathcal{F}_t)_t$, $\langle .,. \rangle$ is the Euclidean scalar product in $\mathbb{R}^d$ and, for all $i = 1, \dots, n$, σ is a d- dimensional process such that $\sigma = (\sigma^i)_i \in \mathcal{A}(\varepsilon)$, where ε is a closed bounded set in the space of $n \times d$ real matrices $M(n, d, \mathbb{R})$ and $\mathcal{A}(\varepsilon)$ (which we call set of admissible volatilities) is the set of ε-valued processes progressively measurable with respect of $(\mathcal{F}_t)_t$.

We can write the dynamics of the risky assets in a none compact vectorial notation in this way:

$$dS_t = \bar{S}_t \sigma_t dW_t$$

$$\bar{S} = diag(S) = \begin{pmatrix} S^i & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & S^i \end{pmatrix}. \qquad (8)$$

In this paper concern is shown mostly to the Generalized Hyperbolic variates. The probability density function of the Generalized hyperbolic distribution (GH_d) is given as(Necula, 2009):

$$P_{GH}(X, \lambda, \alpha, \beta, \delta, \mu) = a(\lambda, \alpha, \beta, \delta, \mu) \left((\delta^2 + \{x - \mu\}^2) \right)^{\frac{1}{2}\lambda - 1/4} \times$$
$$B\left(\lambda - 0.5, \alpha\sqrt{\delta^2 + x^2 - 2x\mu + \mu^2}\right) e^{\beta(x-\mu)} \qquad (9)$$

where $a(\lambda, \alpha, \beta, \delta, \mu) = \frac{(\alpha^2 - \beta^2)^{\frac{1}{2}\lambda}}{\sqrt{2\pi}\alpha^{\lambda - \frac{1}{2}}\delta^\lambda B\left[\lambda, \delta\sqrt{\alpha^2 - \beta^2}\right]}$ and $B(\lambda, .)$ denotes the modified Bessel function of the third kind with index λ. For $\lambda = 1$, the Generalized Hyperbolic Distribution is called the simple Hyperbolic Distribution $H(\alpha, \beta, \delta, \mu)$.

$$P_H\ (X,\alpha,\beta,\delta,\mu) = \frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta\beta\left(1,\delta\sqrt{\alpha^2-\beta^2}\right)} exp\left(-\alpha\sqrt{\delta^2+\{x-\mu\}^2}+\beta(x-\mu)\right). \qquad (10)$$

The name of hyperbolic distribution derives from the fact that the log-pdf represents the equation of a hyperbola.

Considering an asset dynamics given by the exponential Brownian motion process, we can find the price of a European call option strike price K at exercise time T as (Benth et al, 2005);

$$X_0 = e^{-rt}E_Q[\max(S(T)-K,0)], \qquad (11)$$

where r is the risk-free interest rate and Q is an equivalent martingale measure. Consider a standard Black-Scholes type market consisting of one risk-free bond and N risky stocks and a finite time horizon $[0,T]$(Korn, 1997). The price process of underlying assets $S(t)$ is governed by the following stochastic differential equation

$$dS(t) = S(t)\mu dt + \sigma S(t)dW_t, \qquad (12)$$

where μ and σ are the drift and the diffusion of the asset value, and W_t is a standard Brownian motion. The institution hedges the asset's value using put options. Let today's market price be defined of a τ-period put as $X(t) = X(s(t),k,r,\tau,\sigma)$ (Huang, et al, 2012). For simplicity, we assume that all options are priced according to the Black-Scholes option pricing models is(Merton,et al 1978);

$$X(t) = Ke^{-\tau r}\phi(d_1) - s(t)\phi(d_2), \qquad (13)$$

where

$$d_1 = \frac{\log\frac{K}{s}-\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = \frac{\log\frac{K}{s}-\left(r+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}.$$

2.1. Linear Transformation

Let $X_1,\dots,X_m$ be m independent GH$_d$ variable with common shape parameters α,β,γ, but having individual location parameters $\mu_1,\dots,\mu_m$ and individual scale parameter $\delta_1,\dots,\delta_m$. Then, the sum variable $Y = X_1+\cdots+X_m$ is also GH$_d$. That is $X\sim GH_d(\lambda,\chi,\psi,\mu,\varepsilon,\gamma)$ and $Y = AX+b$ where $A\in R^{k\times d}$ and $b\in R^k$, then

$Y \sim GH_k(\lambda, \chi, \psi, A\mu + b, A\varepsilon A', A\gamma)$.

Let $\chi \sim Skewed\ T_d(\sigma, \mu, \varepsilon, \gamma)$and $Y = AX + b$, then

$Y \sim skewed\ T_K(\sigma, A\mu + b, A\varepsilon A', A\gamma)$ (Hu and Kercheval, 2007).

Now if $A = w^T = (w_1, \dots, w_d)$ and $b = 0$, then the portfolio $y = w^T X$ is a one dimensional skewed-t distribution and

$$Y \sim skewed\ T_I(\sigma, w^T\mu, w^T\varepsilon w, w^T y). \qquad (14)$$

Thus the marginal distribution are automatically obtained once the multi-variance distributions are calculated, i.e, $\chi_i \sim skewed\ T_I(\lambda, \chi, \psi, \mu_i, \varepsilon_{ii}, \gamma_i)$. Here skewed-t distribution is subfamily of generalized distribution(GH_d) championed by Mc Neil et al (2005). GH_d with the parameterization above is closed under linear transformation. Under the $GH_d, X = \frac{S(t+\Delta t) - S(t)}{S(t)}$ is the asset price return at a given time scale T, and $S(t)$ is the asset price at time t.

2.2.The Optimization Model

Under the assumption that the compound return is distributed according to GH, given the linear transformation, the optimization problem is formulated into equivalent minimization problem.

$$\min\ Ax + \partial\psi(x) \ni b \qquad (15)$$

$$S.t.\ \begin{Bmatrix} Ax + b \geq 0 \\ x \qquad \geq 0 \end{Bmatrix}.$$

Where $A = \{a_{ij}\}$ is a given $n \times n$ real matrix and b is a given n-dimensional space. φ is a convex function not necessarily differentiable, and it is well known that if $D(\varphi) = \{x \in R^n : \varphi(x) < \infty\} \neq \emptyset, then\ for\ x \in D(\varphi)$ the sub gradient $\partial\varphi$ of $\varphi: R^n \to R$ at x is defined as (Okoroafor and Osu ,2004)

$$\partial\varphi(x) = \{g \in R^n : f(x + t) - f(x) \geq \langle g, t \rangle\}\ \forall\ x + t \in D(\varphi) \qquad (16)$$

And it is a monotone. R^n a Euclidean n-dimensional space with the usual norm $\|x\|^2 = x'x$ and inner product $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, where x' denotes the transpose of the vector $x \in R^n$.

Equation (15) is a special case of a generalized equation consisting typically of a smooth part H_1(the gradient of a real valued differentiable convex function) and a multivalued non-smooth part H_2(the sub-gradient of a proper lower semi continuous function) expressed in the form

$$H_1(x) + H_2(x) + b'x, \quad (17)$$

So that the problem (15) is equivalent to minimizing the function f defined as

$$f(x) = \frac{1}{2}x'Ax + \varphi(x) - b'(x) \quad (18)$$

a number of procedures are available for solving such problems as in (18) (see for example Uko, (1992)). In what follows, we formulate a stochastic algorithm method since the form of (18) suggests a reformation of the original multi valued problem as a search for zero of a single-valued section of the smooth function ∂f.

3. Formulation of Stochastic Algorithm

Denote $\partial f^k = \frac{\partial f(x^k)}{\partial x}, \frac{\partial^2 f(x^k)}{\partial x_r \, \partial x_s} = \partial^2_{r,s} f^k$

As in (Okoroafor and Osu, 2004), we constructed a sequence of random vectors $d^k \in R^n$ that strongly approximate $\partial f^k = \partial f(x^k)$ for each k in the sense that

$$E\|d_j^k - \partial f^k\| = 0.$$

And their expected Euclidean distance

$$E\|d_j^k - \partial f^k\|^2$$

is minimum so that a search in the direction of the random sequence $\{d_j^k\}$ approximates a search through the true gradient ∂f^k and this is expected to lead to the non-zero global minimizing factor if it exists. To this end, we consider the natural Taylor's expansion of a quadratic function f about point x_0 given by

$$f(x) - f(x_0) = \langle \partial f(x_0), x - x_0 \rangle + \frac{1}{2}(x - x_0)H(x_c)(x - x_0) \quad (19)$$

where x_c is on the line segment between x and x_0 and $H(x_c)$ is the Hessian of f at x_c.

Let $e(x_j)$ be a sequence of non-observable random errors satisfying

$$Ee(x_j) = 0 \text{ for each j}$$

and

$$Ee(x_j)e(x_j) = \sigma^2\delta_{ij}, 0 < \sigma^2 < \infty.$$

Let $y(x_1), y(x_2), \dots, y(x_m)$ be real- valued independent observable random variable performed on $x_{1,}x_2, \dots, x_n$ $n + 2 < m < \frac{1}{2}n(n + 1)$ chosen in the neighbourhood of x^k for a fixed k, the

$$y_j = y(x_j) = f(x + t_j) - f(x_j)$$

$$= \langle\partial f(x^k), t_j\rangle + \frac{1}{2}\sum_{k=1}\sum_{r=1} t_{kj}t_{rj}\, \partial^2_{kr}f^2 + e(x_j) \qquad (20)$$

is identifiable with (15) so the fixed $t_j \in R^n$ satisfying $\sum_{i=1}^m t_{ij} = 0\,, \frac{1}{m}\sum_{i=1}^m t_{ij}{}^2$ linearizes f, (Okoroafor and Osu, 2005) and hence the least square approximation

$$d^k = M^{-1}\sum_{j=1}^m t_j y_j \; M = \sum_{j=1}^m t_j t_j^i\,, \qquad (21)$$

exists and is adequate for approximation ∂f such that Euclidean distance

$$E\|d^k - \partial f(x^k)\| = 0 \text{ for each k} \qquad (22)$$

and also yields, by elementary calculation the minimum Euclidean distance

$$E\|d^k - \partial f(x^k)\| = M^{-1}\sigma^2 \qquad (23)$$

in the sequel we assume, without loss of generality that $\sigma^2 = 1. \{d^k\}$ is, thus, a sequence of independent and identically distributed random vectors and determines the direction of search.

It follows that by letting x^0 be an initial point, the sequence of path produced by $\{x^k\}_{k=0}^\infty$ through its definition

$$x^{k+1} = x^k - \rho^k d^k$$

By successive iteration, is the trajectory of the point x^0 and any limiting point of the sequence is therefore the attractor of x^0.

3.1. Getting The Domain of Attraction

Let $R_+^n - N(0)$ be partitioned into exclusive segments, $S_j, j = 1,2, \dots, t, n < t \le 2^n$.

Let x_j be chosen randomly in S_j, such that $f(x_j) > 0, \forall j$

Let $P_j = P(x_j = \alpha)$be the probability that $x_j = \alpha$ so that

$$P_j \ge 0, \sum_{j=1}^t P_j = 1 \qquad (24)$$

Put

$$\mathrm{P_j} = \frac{\mathrm{f(x_j)}}{\sum_{\mathrm{j=1}}^{\mathrm{t}} \mathrm{f(x_j)}}$$

So that

$$\bar{\mathrm{x}} = \sum_{\mathrm{j=1}}^{\mathrm{t}} \mathrm{x_j P_j} = \sum_{\mathrm{j=1}}^{\mathrm{t}} \frac{\mathrm{x_j f(x_j)}}{\sum_{\mathrm{j=1}}^{\mathrm{t}} \mathrm{f(x_j)}} \ . \qquad (25)$$

It is shown in (Okoroafor and Osu, 2004) that if

$$\hat{\mathrm{x}} = \bar{\mathrm{x}} - \rho \mathrm{d}, \rho > 0 \ , \qquad (26)$$

where d is as in(21), then

$\mathrm{f}(\hat{\mathrm{x}}) = \min\langle \mathrm{f}(\mathrm{x_j}) : \mathrm{x_j} \in \mathrm{s} \rangle$. It follows that the segment $\mathrm{S_T}$ where $\hat{\mathrm{x}} \in \mathrm{S_T}$ contains $\mathrm{x} > 0$ for which $\mathrm{f(x)}$ is minimum and hence we have $\varphi(\mathrm{V_{\bar{x}}}) \subset \mathrm{S_T}$ so that if $\{0\}$ is the attractor of the point $\bar{\mathrm{x}}$ and $\varphi(\{0\}) \cap \varphi(\mathrm{V_{\bar{x}}}) = \emptyset$ then $N(0) \cap N(V_{\hat{x}}) = \emptyset$ or else $N(0) = N(V_{\hat{x}})$ with global domain of attraction $\varphi(0) = \varphi(V_{\bar{x}})$. Where

$$\mathrm{V_{x^*}} = \{\mathrm{x^*} \in \mathrm{R^n} : \mathrm{x^*} > 0 : \partial \mathrm{f(x^*)} = 0\} \qquad (27)$$

is a way of stochastically solving problem (15). Thus we have

Lemma 1: suppose that $\mathrm{V_{\hat{x}}} \neq \phi$. thus there exists a neighborhood $\mathrm{N(V_{\hat{x}})} \subseteq \mathrm{D}(\partial \mathrm{f})$ of $\mathrm{V_{\hat{x}}}$ such that for any initial guess $\hat{\mathrm{x}} \in \varphi(\mathrm{V_{\bar{x}}})$, the non-negative minimizer $\mathrm{V_{\hat{x}}}$ is obtained as the limit of iteratively constructed sequence $\{\mathrm{x^j}\}_{\mathrm{j=1}}^{\infty}$ generated form $\hat{\mathrm{x}}$ by $\mathrm{x^{j+1}} = \mathrm{x^j} - \rho^{\mathrm{j}} \mathrm{d^j}$.

Then with $\hat{\mathrm{x}}$ as our starting point we search for the minimizer of f as follows: starting at $\hat{x}$ as in Eq. (26).

A. Compute the d^k as in Eq. (21)

B. Compute the corresponding ρ as specified below

C. Compute $x^{k+1} = x^k - \rho^k d^k$

Has the process converged? i.e., $\|x^{k+1} - x^k\| < \sigma, \sigma > 0$ if yes, then $x^{k+1} = x^k$, if no return to A.

Here we prove the strong convergence of the sequence to the solution of (27)

Theorem 1: Let $\{\rho^k\}$ be a real sequence such that

I. $\rho^0 = 1, 0 < \rho^k < 1 \forall k > 1$

II. $\sum_{k=0}^{\infty} \rho^k = \infty$

III. $\sum_{k=0}^{\infty} \rho^{2k} < \infty$

Then the sequence $\{x^k\}_{k=0}^{\infty}$ generated by $\hat{x} \in \varphi(V_{\hat{x}}) \subseteq D(\partial f)$ and defined iteratively by $x^{j+1} = x^j - \rho^j d^j$ remain in $D(\partial f)$ and converges strongly to $V_{\hat{x}}$.

Proof: Let $b^k = \rho^k \|d^k - \partial f^k\|$

Then $\{b_k\}_{k+1}^{\infty}$ is a sequence of independent random variable and from (22) $Eb_k = 0$ for each K.

Noticing that the sequence of partial sums $\{S_k\}_{k+1}^{\infty}$, $S_k = \sum_{j=1}^{k} b_j$, is a Martingale. Therefore,

$$ES_k^2 = \sum_{j=1}^{k} Eb_j^2 = \sum_{j=1}^{k} \rho^{2j} E\left\|d^j = \partial f^j\right\|^2$$

$$= M^{-1}\sigma^2 \sum_{j=1}^{k} \rho^{2j}.$$

And

$\sum Eb_j^2 < \infty,\ since\ \ \sum_{j=1}^{k} \rho^{2j} < \infty$

Hence by a version of Martingale convergence theorem (Whittle, 1976), we have

$$\lim_{k \to \infty} S_k = \sum_{j=1}^{\infty} b_j < \infty$$

So that

$$\lim_{k \to \infty} \rho^k = \|d^k - \partial f^k\| = 0$$

Noticing that in (15), A is positive definite so that $f(x)$ is convex and hence ∂f is monotone. But an earlier result in theory of monotone operators, due to (Chidume, 1990), shows that the sequence $\{x^k\}$ generated by $x^0 \in D(\partial f)$ and defined iteratively by:

$$x^{k+1} = x^k - \rho^k \partial f^k$$

Remain in $D(\partial f)$ and converges strongly to $\{x^*: \partial f(x^*) = 0\}$. It follows from this result that our sequence converges strongly to V_{x^*} if $V_{x^*} \neq 0$.

4. Empirical Example

The following illustrates the method in a concrete setting;

$$\begin{pmatrix} 1 & {}^{1}/_{2} & {}^{1}/_{3} \\ {}^{1}/_{3} & 1 & {}^{1}/_{2} \\ {}^{1}/_{2} & {}^{1}/_{3} & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} {}^{11}/_{18} \\ {}^{11}/_{18} \\ {}^{11}/_{18} \end{pmatrix}$$

, the actual solution by Richardson method at the eighty iteration is $X^{*'} = (0.333333 \quad 0.333333 \quad 0.333333)$. The above procedure starting at $X_0^{'} = (0 \quad 0 \quad 0)$ gives after one iteration;

$X^{*'} = (0.333333688 \quad 0.333333688 \quad 0.333333688)$, which is quite close to the solution. This gives the value of $Y^{*'}$ as:

$Y^{*'} = (0.611111111 \quad 0.611111111 \quad 0.611111111)$, and the difference between $Y^{*'}$ and $X^{*'}$ gives $Y^{*'} - X^{*'} = 0.833333333$ (using 7a and 7b).

This shows that the difference between the Geometric returns and Arithmetic returns is not negligible.

4.1. Concluding Remarks

In the present paper, after carefully studying the process with hyperbolic returns, we discovered that, if X_t is small, in practice the volatility of a price series is small, and the time resolution is high, geometric and arithmetic returns are quite similar, but when volatility increase and the time resolution decreases, the difference grows large. We introduce a transformation $Y = A\,x + b$ where $X \sim GH_d(\lambda, \chi, \psi, \mu, \varepsilon, \gamma)$, A is a real $n \times n$ coefficient matrix and b is $n\,x\,1$ dimensional real vector taken Y as the sum of geometric returns and X as the sum of arithmetic returns. The numerical difference between X_t and Y_t is not negligible if the time resolution decreases and the volatility of price series is large.

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