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TRADING ON A MOMENTUM OPPORTUNITY

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Abstract. There is an extensive academic literature that document that assets which have performed well in the past will continue to perform well over some holding period, so called momentum. The momentum effect has been found to disappear with time. In the present paper, the performance of the asset is modelled as a Brownian motion with positive drift, for which the drift turns negative at an unobservable exponentially distributed random time. We investigate how an investor should trade optimally on a momentum opportunity to maximize her expected profit. We show also that the optimal boundary at which the investor should liquidate the trade depends monotonically on some model parameters.

Keywords: free-boundary problems; momentum; monotonicity; optimal stopping theory; trading.

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1. Introduction

Momentum is the notion that an asset which has performed well in the past will continue to do this, and vice versa. This may be the most common asset management strategy, and there is a large literature about it. The pioneering paper [2] documents that stocks with high recent performance continue to generate significant positive returns over a 3-12 months holding period. This conclusion was further established in [3], and [6] found that

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momentum is present also in international markets. These papers also show that the momentum effect decays in time to eventually disappear entirely.

The focus of most studies on momentum is to evaluate whether there are statistically significant momentum effects, but not on how to make optimal trades. In this paper, we seek to maximize the expected profit from a momentum trade. The expected return of an asset is assumed to be constant up until some random, and unobservable, time θ, after which it turns negative. Hence, the momentum effect will eventually disappear. Based on this model for the expected return, we seek the optimal liquidation level. Intuitively, the investor will hold the position as long as the momentum is present. However, if it has disappeared, the investor will want to exit the position, and find some better investment. The problem is to know when to do this. The assumption that the momentum trend eventually disappears is in accordance with the consensus in the financial literature, see e. g. [2], [3], and [6]. In [1], a problem similar to ours is solved, but where the asset is modelled by a geometric Brownian motion, i e the Black-Scholes model.

We show in this note that that the solution to the present problem can be obtained from well-known techniques, see e g [5, Ch 22], by rewriting the optimal stopping problem in a suitable way. It is given as the explicit solution to a free-boundary ordinary differential equation. Further, the stopping region is a constant barrier in terms of the conditional probability that $\theta \leq t$. We establish also the monotonicity of the stopping boundary with respect to the model parameters.

We define the asset price model in Section 2. In Section 3, the optimal liquidation problem is set up and a solution is derived.

2. The model

The benchmark relative asset price process is assumed to be a Brownian motion with drift, which changes from $\mu > 0$ to $-r$, $r > 0$ at some random time θ, which takes value 0 with probability π, and is exponentially distributed with parameter $\lambda > 0$ conditional on $\theta > 0$.

We take as given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Further, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion independent of θ, which starts at 0.

We observe the asset price process $X = (X_t)_{t\geq 0}$ with dynamics

$$dX_t = ((\mu + r)I(t \leq \theta) - r)dt + \sigma dW_t,$$

where $X_0 = 0$. Hence, we have that

$$X_t = \begin{cases} \mu t + \sigma W_t & \text{if } t < \theta \\ (\mu + r)\theta - rt + \sigma W_t & \text{if } t \geq \theta, \end{cases}$$

where $\mu, \sigma, r > 0$ are given. Denote by $\mathcal{F}^X$ the filtration $\mathcal{F}^X_t = \sigma\{X_s : 0 \leq s \leq t\}$, and note that $\mathcal{F}^X$ is strictly smaller than $\mathcal{F}$.

The intuition behind this model is that the investor can observe an asset price which appears to have a distinctly positive drift. However, at some unknown time point, which might already have passed when the investor enters the trade, the drift becomes negative.

When the drift turns negative, the investor wants to liquidate the trade and find another investment. The problem of finding the optimal liquidation strategy is the topic of the next section.

3. The optimization problem

In this section, we find the optimal stopping time for the investor to liquidate the trade. This is done by defining a value function, which we maximize. Mathematically, this amounts to solving a free-boundary problem to find a candidate solution. We can then verify that this solution coincides with the optimal value function.

We denote by Λ the set of $\mathcal{F}^X$-stopping times. Given the asset price model introduced in the previous section, we seek a $\tau_* \in \Lambda$ such that we obtain the optimal value function

$$V(\pi) = \sup_{\tau \in \Lambda} \mathbb{E}[X_\tau]. \tag{3.1}$$

The function V states simply that the investor wants to maximize her profit from the trade.

We define now the *a posteriori probability process* $\pi_t = \mathbb{P}(\theta \leq t|\mathcal{F}_t^X)$ for $t \geq 0$ with $\pi_0 = \pi$. It is well-known from filtering theory, see e g [7], that the dynamics of $(\pi_t)_{t\geq 0}$ satisfies

$$d\pi_t = \lambda(1-\pi_t)dt + \frac{\mu+r}{\sigma}\pi_t(1-\pi_t)d\bar{W}_t,$$

for the process

$$\bar{W}_t = \frac{1}{\sigma}\left(\int_0^t \mu - (\mu+r)\pi_s ds - X_t\right), \tag{3.2}$$

which is a standard Brownian motion with respect to $\mathcal{F}_t^X$, see [4, Ch. 4]. Note again that the process $(\pi_t)_{t\geq 0}$ is conditioned only on the σ-algebra generated by the asset price process, and not the larger filtration $\sigma\{(X_s)_{0\leq s\leq t}, \theta\}$. Hence, the dynamics of the probability process $(\pi_t)_{t\geq 0}$ are in terms of the asset price process only.

In view of Equation (3.2), one can reformulate the optimal value function in Equation (3.1) to obtain

$$V(\pi) = \sup_{\tau\in\Lambda} \mathbb{E}\left[\int_0^\tau (\mu - (\mu+r)\pi_t)dt\right]. \tag{3.3}$$

This problem is similar to known results, e g [5, Ch. 22], and for this reason we are brief in the derivation of the solution below.

Standard optimal stopping theory suggests that the optimal stopping time τ_A is of the form

$$\tau_A = \inf\{t \geq 0 : \pi_t \geq A,\ A \in [0,1]\},$$

and that (A, V) are given as the solution to the free-boundary problem

$$\begin{cases} \mathbb{L}V = -\mu + (\mu+r)\pi & \text{for } \pi \in (0, A) \\ V(A) = 0 & \\ V'(A) = 0 & \\ V(\pi) > 0 & \text{for } \pi \in (0, A) \\ V(\pi) = 0 & \text{for } \pi \in (A, 1], \end{cases} \tag{3.4}$$

where

$$\mathbb{L} = \lambda(1-\pi)\frac{\partial}{\partial\pi} + \frac{(\mu+r)^2}{2\sigma^2}\pi^2(1-\pi)^2\frac{\partial^2}{\partial\pi^2}.$$

The ordinary differential equation in Equation (3.4) has the general solution

$$V(\pi) = C_1 + \int_0^{\pi} e^{-\frac{\lambda}{\gamma} f(x)} \left(C_2 - \frac{1}{\gamma} \int_0^x \frac{e^{\frac{\lambda}{\gamma} f(y)} (\mu - (\mu + r) y)}{y^2 (1-y)^2} dy \right) dx,$$

where $f(x) = \log(\frac{x}{1-x}) - \frac{1}{x}$, $\gamma = (\mu + r)^2/(2\sigma^2)$, and C_1, C_2 are constants to be determined. Inserting this solution into the free boundary problem (3.4) yields the equation

$$\int_0^{A_*} \frac{e^{\alpha f(x)} (\beta - x)}{x^2 (1-x)^2} dx = 0, \tag{3.5}$$

for the optimal liquidation boundary A_*, where $\alpha = \frac{\lambda}{\gamma} > 0$, and $\beta = \mu / (\mu + r) > 0$.

The proof of the following lemma is straightforward, and we omit it.

Lemma 3.1. *Equation (3.5) admits a unique solution* $\frac{\mu}{\mu + r} < A_* < 1$.

Given A_*, our candidate solution to the optimal stopping problem (3.4) becomes

$$V_*(\pi) = \int_{\pi}^{A_*} e^{-\frac{\lambda}{\gamma} f(x)} \frac{1}{\gamma} \int_0^x \frac{e^{\frac{\lambda}{\gamma} f(y)} (\mu - (\mu + r) y)}{y^2 (1-y)^2} dy dx, \tag{3.6}$$

for the optimal stopping time

$$\tau_{A_*} = \inf\{t \geq 0 : \pi_t \geq A_*\}. \tag{3.7}$$

We present now a verification theorem stating that the candidate solution derived above indeed coincides with the optimal value function $V(\pi)$ in (3.3). The proof is completely analogous to [5, Thm. 22.1] and we omit it.

Theorem 3.2 (Verification theorem). *The value function $V(\pi)$ in (3.3) is given by Equation (3.6). Further, the optimal stopping time τ_{A_*} is given by Equation (3.7).*

We analyze now how the optimal liquidation boundary A_* depends on the model parameters. It is shown that the parameters μ, σ, and λ alter A_* monotonically. We will need the following simple lemma in the proof.

Lemma 3.3. *Let $0 < \beta < A$. Assume that a measurable function $f < 0$ on $(0, \beta)$, $f > 0$ on (β, A), and satisfies $\int_0^A f(x)\, dx = 0$. Further, the measurable function g is increasing. Then $\int_0^A f(x)\, g(x)\, dx \geq 0$.*

Proof. We define the function $\tilde{g} = g + C$, for some constant C, such that $\tilde{g}(\beta) = 0$. Then

$$\int_0^A f(x)\, g(x)\, dx = \int_0^A f(x)\, \tilde{g}(x)\, dx - C \int_0^A f(x)\, dx$$
$$= \int_0^\beta f(x)\, \tilde{g}(x)\, dx + \int_\beta^A f(x)\, \tilde{g}(x)\, dx \geq 0.$$

Proposition 3.4. *The optimal liquidation boundary A_* is monotonically increasing in the parameter μ, and decreasing in σ and λ.*

Proof. We define the function

$$g(A_*, \mu, r, \sigma, \lambda) := \int_0^{A_*} \frac{e^{\frac{\lambda 2\sigma^2}{(\mu+r)^2} f(x)} (\mu - (\mu + r)\, x)}{x^2(1-x)^2} dx,$$

and let g_i' denote the differential with respect to the ith argument. The proof consists of applying the implicit function theorem to the relation $g(A_*, \mu, r, \sigma, \lambda) = 0$. We see that

$$g_1'(A_*, \mu, r, \sigma, \lambda) < 0,$$

since $A_* > \beta$ by Lemma 3.1. The fact that $g_2' \geq 0$ follows from Lemma 3.3. Finally, $g_4', g_5' \leq 0$ are immediate from Lemma 3.3. The implicit function theorem gives the results.

It can be verified that we have no monotonicity of the optimal liquidation boundary with respect to the parameter r.

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