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EVALUATION OF ENERGY FORWARD CURVES WITH JUMPS UNDER THE GENERAL LÉVY PROCESS

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Abstract. In this study, we evaluate the relationship between the forward rates and the future delivery period with the consideration of the Lévy process for a time-inhomogeneous exponential jump-diffusion process and model the forward curve. This is a large variety of stylized features observed in the Samuelson effect of increasing volatilities close to maturity. However, a new method based on characteristic functions is used to estimate the jump component in a finite-activity Lévy process, which includes the jump frequency and the jump size distribution which enables the further investigation of the properties of estimators without the presence of high frequency data Δ. Numerical implementation of the approach was applied on sample electricity data of about 10,000 observations between the period of 2 years and then a seasonalized forecast for an extra year was implemented to normalize the volatility in forwards contracts.

Keywords: futures; forwards; contract; volatility; Lévy process; energy; jumps.

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1. INTRODUCTION

The evaluation of high frequency financial rates with time is paramount for investors as this influence trading strategies and investment in futures and forwards contracts. Also, understanding the concepts of futures and forwards contracts is key for decision making as it concerns which contract enhances profitability in the options market. While the former is a commitment to trade a specified quantity and quality of a particular asset at a specified price to be delivered on a specified future date, the later is a direct agreement between two parties made over the counter on fixed terms at a future date. Essentially, futures allows the investor to "lock in" a price now so as to potentially benefit in future if prices fluctuates. The price of the asset is agreed upon at the time the commitment is made. The long party, who is the buyer, agrees to receive the underlying commodity. Futures are standardized, in terms of dates, quality and amounts traded, and can be re-traded during their life time on a futures exchange.

Although forwards and futures all involve an agreement to trade at a certain future date for a certain price, they have significant differences. Foremost, futures are standardized meaning they are exchange traded, whereas forwards trade between individuals institutions (Over-the-counter). Secondly, cash flows of the two contracts occur at different times. Forwards are settled once at maturity whereas futures are marked to market daily with cash-flows exchanging between the long and short positions to reflect the daily price variations. However, if future interest rates are known with certainty then both forwards and futures can be treated as similar for purpose of pricing.

Meanwhile, the term structures (i.e. forward curve) which is a function graph in finance that defines the prices at which a contract for future delivery or payment can be concluded today in the markets is one of those signals that offer a considerable amount of information as to the market sentiment and the potential direction of the market. The term forward curve refers to a series of consecutive month's prices for future delivery of an asset - like West Texas Intermediate (WTI) or any of the main energy products traded on New York Mercantile Exchange (NYMEX). Thus the forward curve of the market is in essence a model showing how future months are valued relative to the nearby or spot contract month given all of the available market information at any instant in time. The shape of the forward curve is important to energy market participants.

The forward curve of the forward market is looking at prices from many different maturities as they extend into the future.

However, a forward interest rate is a type of interest rate that is specified for a loan that will occur at a specified future date. As with current interest rates, forward interest rates include a term structure which shows the different forward rates offered to loans at different maturities. The shape of the futures curve is important to commodity hedgers and speculators. Both care about whether commodity futures markets are contango or normal backwardation markets. These two terms refer to the pattern of prices over time, specifically if the price of the contract is rising or falling. While **Contango** implies futures prices are falling over time as new information brings them into line with the expected future spot price while the **Normal backwardation** is when the futures price is below the expected future spot price. This is desirable for speculators who are net long in their positions: they want the futures price to increase.

Although in practice the futures and forwards contracts are seen as different entities [1] emphasized that due to the nature of the energy commodity market, the terms forward and futures price can be used interchangeably because they represent the same value if a technical condition, namely that the delivery and the payment dates of both contracts coincide and there is no possibility of default on either side, is satisfied. Hence, the term forward curve is often used to graphically represent both forward and futures prices in relation to the maturation time. Therefore, for the purpose of justification, we introduce the following proposition.

Proposition 1. *Let $X = (X_t)_t$ be any real-valued discrete process on a probability space $(\Omega, F, \mathbb{P})$ representing the pricing decisions by a potential investor as described on a curve (forward or future) in discrete-time. For each discrete-time $t \geq 0$, we define $F_t^X = \sigma(X_0, .X_1, ..., X_t)$ and $F_{t+1}^X = \sigma(X_0, .X_1, ..., X_{t+1})$, the futures and forward contracts respectively as σ-algebra generated by random variables $X_0, X_1, ..., X_t$ which models the information about values of the price process X_t up to time t. Then, for the purpose of this study if we assume that futures curves are submerged in forwards curves in discrete time then the following assumptions are true;*

(1) The sequence of futures contracts $\mathbb{F}^X = \left(F_t^X\right)_t$ is a discrete-time filtration called the neutral filtration of the process X_t which is neutralized by the sequence of forwards contracts $\left(F_{t+1}^X\right)_t$ also with a filtration generated by X_{t+1}.

(2) *Moreover, the discrete-time processes X_t and X_{t+1} are adapted to their natural filtration $\mathbb{F}^X$ such that;*

$$F_t^X \subset F_{t+1}^X \tag{1}$$

Proof: $\forall t$, *X_t is F_t^X-measurable where*

$$F_t^X = \sigma(X_0, X_1, ..., X_t)$$
$$= \sigma\left(\bigcup_{n=0}^{t} \sigma(X_n)\right)$$

Also, $\forall t$

$$F_t^X \subset F_{t+1}^X$$

Let

$$F_{t+1}^X = \sigma(X_0, X_1, ..., X_t, X_{t+1})$$

and

$$F_{t+1}^X = \sigma\left(\bigcup_{n=0}^{t+1} \sigma(X_n)\right)$$
$$F_t^X = \sigma\left(\bigcup_{n=0}^{t} \sigma(X_n)\right)$$

however,

$$F_{t+1}^X = \sigma\left(\bigcup_{n=0}^{t+1} \sigma(X_n)\right) = \left(\sigma\left(\bigcup_{n=t+1} \sigma(X_n)\right) \bigcup \sigma\left(\bigcup_{n=0}^{t} \sigma(X_n)\right)\right)$$
$$= \sigma\left(\bigcup_{n=t+1} (X_n)\right) \bigcup F_t^X$$
$$= \sigma(\sigma(X_{n+1})) \bigcup F_t^X$$

such that

$$F_{t+1}^X = \sigma(\sigma(X_{n+1})) \bigcup F_t^X \supseteq F_t^X$$

and

$$F_t^X \subset \sigma(\sigma(X_{n+1})) \bigcup F_t^X$$

Hence it is true that

$$F_t^X \subset F_{t+1}^X$$

Therefore by Equation (1), it is convenient to use forward contracts (forward curves) in this study, as that will also imply the futures contracts (futures curves) accordingly without loose of generality. In literature, although pricing futures and forwards appears to be the same on inception and at maturity of the spot market, however pricing forwards is sufficiently more convenient as a benchmark for pricing energy derivatives [2].

2. Preliminaries

2.1. Forward Curve Modelling and Representation. Evaluating the energy forward curve requires a proper model framework to benchmark the jumps in the curves. As a result, we are interested in studying the dynamics of the forward curve $F(t,T)$, $0 \leq t \leq T$, of a contract delivering a commodity at time $T > 0$, and adopting the Heath-Jarrow-Merton (HJM) approach gives the dynamics of F as the solution of a stochastic differential equation

$$dF(t,T) = \alpha(t,T)dt + \sigma(t,T)dL(t) \tag{2}$$

for some infinite-dimensional Lévy process $L(t)$ and appropriately defined parameters α and σ. Since the forward curve $F(t,T)_{T\geq t}$ changes its domain over time is more convenient to work with the Musiela parametrisation introduced by [3];

$$f(t,\tau) = F(t,t+\tau), \tau \geq 0. \tag{3}$$

We interpret $\tau = T-t$ as time to delivery, whereas T is the time of delivery and $T = \tau + t$. Then heuristically speaking, the forward curve follows the stochastic partial differential equation (SPDE)

$$f(t,T-t) = F(t,t+T-t) = F(t,T)$$

which is replaced in Equation (2) such that

$$dF(t,T) = df(t,T-t) = \alpha(t,T)dt + \sigma(t,T)dL(t)$$

$$df(t,T-t) = \alpha(t,T)dt + \sigma(t,T)dL(t)$$

$$df(t,\tau) = (\alpha(t,t+\tau) + \alpha_0 f(t,\tau))dt + \sigma(t,t+\tau)dL(t). \tag{4}$$

However, in order to make sense of this SPDE, it is useful to work in an appropriate space of function which contain the entire forward curve $(f(t,\tau))_{\tau\geq 0}$ for any time $t \geq 0$. For this purpose, we shall use the spaces of forward curves introduced by [4] such that whenever an arbitrary constant $\beta > 0$, and H_β be a space of all absolutely continuous functions $h : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that;

$$\|h\|_\beta = \left(|h(0)|^2 + \int_{\mathbb{R}_+} \left|h^{'}(\tau)\right|^2 e^{\beta\tau} d\tau\right)^{\frac{1}{2}} < \infty.$$

Since forward curves should flatten for larger time to maturity τ, the choice of the space H_β is reasonable from an economic point of view. We shall mostly be concerned with the more convenient representation of F in terms of the Musiela parametrisation where we get

$$F(t,T) = f(t,T-t),$$

for a function $f(t,\tau), \tau \geq 0$, denoting the forward price at time t for a constant with time delivery τ. We shall interpret f as a stochastic process with values in a space of functions on $\mathbb{R}_+$. Most specifically, we consider the Hilbert space H_G introduced by [4]. These Hilbert-spaces are suitable for modelling the forward rate curves in the Museila parametrisation [4].As the modelling of forward prices are similar in idea to the context of forward rates in fixed-income markets, we adopt the spaces H_G for our analysis. If we model the forward curve directly under the pricing measure $\mathbb{Q}$ which is the common strategy in the HJM approach, martingale conditions must be imposed in the dynamics. It is to be noted that the forward contracts are liquidly traded financial assets in most commodity markets, and therefore, it may sometimes be natural to model the dynamics under the market probability $\mathbb{P}$. In power and gas, and in fact also weather derivative markets, the forward contracts are typically settled over a delivery period and not at a specific delivery time. For example, in the NordPool market, forward contracts are settled financially on the hourly spot price over different delivery periods that range from a day to a year. As an application of our study of operators, we identify forward contracts with a delivery period as an operator on elements in the space defined by [4] mapping onto itself. Hence, the analytic properties of a fixed-delivery forward curve can be transported to a delivery-period forward curve by an operator, being in fact a sum of the identity operator and an explicitly given integral operator.

2.2. The Term Structure of Bond Markets. The most basic interest rate contract is a bond which pays the holder with certainty one unit of cash at a fixed future maturity date T with a price denoted by $P(t,T)$ at time $t \leq T$ where t is any present time before maturation. Therefore, the term structure of bond prices $\{P(t,T)|T \geq t\}$ is a deterministic non-increasing, positive curve with $P(t,t) = 1$. Whereas for fixed maturity T, $\{P(t,T)|t \in [0,T]\}$ is a stochastic process because the economy and the market beliefs about the future value of money, changes in time and is not certain in future. Therefore, a more informative measure of the current bond market at time t is defined as the term structure of interest rates or forward curve $\{f(t,T)|T \geq t\}$ given by

$$P(t,T) = exp\left(-\int_0^{T-t} r(t,s)ds\right) = exp\left(-\int_t^T f(t,s)ds\right), 0 \geq t \geq T \tag{5}$$

The function $f(t,\cdot)$ is a fortiori local integrable and $P(t,\cdot)$ is absolutely continuous. Hence, for the purpose of this study we call $f(t,T)$ the continuously compounded instantaneous forward rate for date T or an interest rate over the infinitesimal time interval $[T, T+dT]$ from time t. Therefore, Equation (5) shows that the forward curve contains all the original bond price information which can be completely recovered. And describing the bond prices in a complete market of a deterministic arbitrage free situation, we have that;

$$P(t,T) = P(t,S)P(S,T), \forall t \leq S \leq T \tag{6}$$

However, if $P(t,T) > P(t,S)P(S,T)$ for some $t \leq S \leq T$, then an arbitrage opportunity is described when at maturity T a bond bought $P(S,T)$ with maturity S at time t leads to a risk-less net gain at terminal time T. And as for arbitrage, it can be formulated such that a complete probability space $\left(\Omega, F, \{F_t\}_{t\in\mathbb{R}_+}, \mathbb{P}\right)$ satisfy the usual conditions, and under such appropriate assumptions, a particular no-arbitrage condition is equivalent to the existence of a probability measure $\mathbb{Q} \sim \mathbb{P}$ under which discounted bond price processes

$$P(t,T)/B(t), t \in [0,T],$$

follow local martingales. Here, we define the amount of cash accumulated $B(t)$ up to time t starting with one unit at time 0 and continually reinvesting at the short $R(s), s \in [0,t]$ by;

$$B(t) = exp\left(\int_0^t R(s)ds\right), t \in \mathbb{R}_+$$

where $R(s) = f(t,t)$. The measure $\mathbb{Q}$ is called an equivalent local martingale measure (ELMM) or risk neutral measure and by the absence of arbitrage we shall mean from now on the existence of such measure $\mathbb{Q}$. Assuming a frictionless market, i.e. there are no transaction costs, taxes, or short sale restrictions which we may have on an investment over the infinitesimal interval $[t, t+dt]$ if the contract is made at t, and the bonds are perfectly divisible then viewing the forwards rates $f(t,T)$ as an estimate of the future short rate $r(t, T-t)$ and assuming $m(t,U) = U-t$, that we had

$$L(t,T,U) = \left(\frac{P(t,T)}{P(t,U)} - 1\right)\frac{1}{m(T,U)} = -\frac{1}{P(t,U)}\left(\frac{P(t,U) - P(t,T)}{U-T}\right) \tag{7}$$

and when we assume U and T to be very close, then

$$\begin{aligned} \lim_{U \longrightarrow T} L(t,T,U) &= \lim_{U \longrightarrow T} \frac{-1}{P(t,U)}\left(\frac{P(t,U) - P(t,T)}{U-T}\right) \\ &= \frac{-1}{P(t,U)}\frac{\partial P(t,T)}{\partial T} \end{aligned} \tag{8}$$

because

$$\frac{\partial \log P(t,T)}{\partial T} = \frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T}$$

We therefore define the instantaneous forward rate with maturity T

$$f(t,T) = -\frac{\delta \log P(t,T)}{\delta T} \tag{9}$$

and under the martingale measure $\mathbb{Q}$, the dynamics of the forward rates are of the Heath-Jarrow-Morton (HJM) form;

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dL(t) \tag{10}$$

where $df(t,T)$ is the instantaneous forward interest rate of zero-coupon bond with maturity T, and it is assumed to satisfy the stochastic differential equation (10), L is an m-dimensional $\mathbb{Q}$-Wiener process (Brownian motion) under the risk-neutral assumption, and the adapted α and σ are the drift and volatility functions respectively, which is investigated in subsequent sections of this study. We take into consideration that the use of the HJM model here is best for use in modeling forward interest rates which are then modeled to an existing term structure of interest rates which is used to determine appropriate prices for interest rate sensitive securities as well as to seek arbitrage opportunities in pricing the underlying contracts. Also, for each maturity date T, the evolution across calendar time t (where $t \leq T$) of the forward rate $f(t,T)$ is govern by (10).

2.3. Forward Curve and Factor Models. In practice, the forward curve cannot be observed directly on the market, hence it has to be estimated and common methods of estimating forward rates curves can be represented as a parametrized family on a forward curve manifold ρ given by;

$$\rho = \{G(\cdot,z) \in C[0,\infty) | z \in Z\} \tag{11}$$

where

$$G : Z \longrightarrow C[0,\infty) \tag{12}$$

is smooth curves with $Z \subset \mathbb{R}^m$ is a finite dimensional parameter set, for some $m \in \mathbb{N}$ and for each appropriate choice of parameter $z \in Z$ [4]. Hence, by slight change in notation, we write the optimal fit of the forward curve as

$$\tau \longrightarrow G(\tau;z) \tag{13}$$

where the variable τ is interpreted as the time to maturity, as opposed to the time **of** maturity T, i.e. $\tau = T - t$. The main problem is to determine under which conditions the interest rate model (10) is consistent with the parametrized family of forward curves (12), in the following sense:

(1) Assume that at an arbitrage chosen time $t = s$, we have fitted a forward curve G to market data. Technically, this means that we have specified an initial forward curve, i.e. for some $z_0 \in Z$ we have

$$f^*(s, s+\tau) = F(\tau, z_0), \forall \tau \geq 0. \tag{14}$$

(2) Is it then the case that the subsequent forward curves produced by the interest rate model (10) always stay within the given forward curve family, i.e. does there at every fixed time $t \geq s$ exist some $z \in Z$ such that

$$f(t,t+\tau) = G(\tau;z), \forall \tau \geq 0? \tag{15}$$

Here, z may depend on t and on the elementary outcome $\omega \in \Omega$.

Now, suppose we introduce the Musiela parametrization $r_t(\tau) = f(t,t+\tau)$ where $\tau \geq 0$. Let $\{S(t)|t \in \mathbb{R}_+\}$ denote the semigroup of right shifts which is defined by

$$S(t)f(\tau) = f(t+\tau)$$

We denote the induced dynamics for the r-process by

$$dr(t,\tau) = \beta(t,\tau)dt + \sigma_0(t,\tau)dL(t), \tag{16}$$

and it is easy to see that there is a one-to-one correspondence between the formulations (10) and (16), namely

$$\beta(t,\tau) = \frac{\partial}{\partial x} r(t,\tau) + \alpha(t,t+\tau) \text{ and}$$

$$\sigma_0(t,\tau) = \sigma(t,t+\tau)$$

Therefore, we can now transfer the HJM drift condition to the Musiela parameterization case.

Proposition 2. *[5] Under the martingale measure $\mathbb{Q}$, the r-dynamics must be of the form*

$$dr(t,\tau) = \left\{ \frac{\partial}{\partial x} r(t,\tau) + \sigma_0(t,\tau) \int_0^\tau \sigma_0(t,z)' dz \right\} dt + \sigma_0(t,\tau)dL(t). \tag{17}$$

Thus the interest rate model is completely characterized by the initial curve and the volatility structure $\sigma_0(t,\tau)$ and hence with the presence of arbitrage situation with a martingale measure $\mathbb{Q}$, it clearly shows that the forward curves is being modelled in an incomplete market. However, as the focus of this current study, we make efforts to integrate the properties of completeness from Banach space in order to evaluate the forward rates in an arbitrage situation as well as time variation which is key while determining the volatility structure (jump diffusion) part $\{\sigma_0(t,\tau)dL(t) : t \geq 0, \tau \geq 0\}$ in Equation (17) and evaluating the size and impact of the jumps as captured by the Lévy process $(L(t))$. In that case, according to [6], the forward rate process $\{r(t,\tau)dL(t) : t \geq 0, \tau \geq 0\}$ must satisfy the following stochastic partial differential equation

$$dr(t,\tau) = \frac{\partial}{\partial x}\left(\left(r(t,\tau) + \frac{1}{2}|\sigma_0(t,\tau)|^2\right)dt + \sigma(t,\tau)\cdot dL(t)\right) \tag{18}$$

for all $t,\tau \geq 0$, where $L(t)$ is also define with the properties of a d-dimensional Brownian Motion, the volatility process $\{\sigma(t,\tau); t \geq 0\}$ is F_t-adapted with values in $\mathbb{R}^d$, while $||$ and $\cdot$ is the standard norm and inner product in $\mathbb{R}^d$, respectively. It is significant to say that (18) is sufficient for the non-arbitrage condition of a complete market which is a paramount characterization for this study.

2.4. Forward Curves with Lévy Jumps. In an incomplete market, energy commodity valuation is best described as stochastic process due to the volatile nature and its subsequent jumps in the forward curves because of the randomness in the pricing models. Hence, motivated by the approach of [7], the jumps in energy forwards is captured as an entity that is best described in a probability space of some adapted process hence;

Definition 1 (Stochastic Process). *A stochastic process X is as a collection of random variables $\{X_t\}_{0\leq t<\infty}$ on a common probability space $(\Omega, F, \mathbb{P})$, where Ω refers to the sample space containing the sequence of energy prices in discrete time, F is a σ-algebra, and $\mathbb{P}$ is a probability measure and the random variables, indexed by some set time T, all take values in the same mathematical space S, which must be measurable with respect to some σ-algebra $\sum$.*

Definition 2 (Brownian Motion). *Standard Brownian motion $W = (W_t)_{0\leq t<\infty}$ satisfies the following three properties;*

(1) $W_0 = 0$
(2) *W has independent increments: $W_t - W_s$ is independent of F_s,* $0 \leq s < t < \infty$
(3) *$W_t - W_s$ is a Gaussian random variable: $W_t - W_s \sim N(0, t-s) \forall 0 \leq s < t < \infty$*

Property (2) implies that Markov property (i.e. conditional probability distribution of futures states depend only on the present state). Property (3) indicates that knowing the distribution of W_t for $t \leq \tau$ provides no predictive information about the process when $t > \tau$.

Definition 3 (**Poisson Process**). *A Poisson process $N = (N_t)_{0\leq t<\infty}$ satisfies the following three properties;*

(1) $N_0 = 0$
(2) N has independent increments: $N_t - N_s$ *is independent of* F_s, $0 \leq s < t < \infty$
(3) N has stationary increments: $P(N_t - N_s \leq \tau)\forall 0 \leq s < t < \infty$

In practice, stochastic differential equations formulated with only Brownian motion or the Poisson process may be ineffective in describing the complex dynamics such as estimating jumps in forward curves. Therefore, the observable jumps in the forward curves of energy market follows a continuous sample path of stochastic processes, however, this study focuses on the discritization of the sample paths and hence we study the price volatility in a filtered probability space projected on some Hilbert space and this motivates the following definition;

Definition 4 (**Lévy Process**). *Let $(\Omega, F, \{F_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where $\mathbb{P}$ denotes the risk-neutral probability. An F_t-adapted process $\{L(t)\}_{t\geq 0} \subset \mathbb{R}$ with $L_0 = 0$ (almost surely) is called a Lévy process if L_t is continuous in probability and has stationary, independent increments with the following properties;*

(1) $\mathbb{P}\{L_0 = 0\} = 1$.
(2) Stationary increments: the distribution $L_{t+s} - L_t$ over the interval $[t, t+s]$ does not depend on t but on length of interval s.
(3) Independent increments: for every increasing sequence of times $t_0, \ldots, t_n$ the random variables: $L_{t_0}, L_{t_1} - L_{t_0}, \ldots, L_{t_{n+1}} - L_{t_n}$ are independent.
(4) Stochastic continuity: $\forall \varepsilon > 0$, $\lim_{h\longrightarrow 0} \mathbb{P}(|L_{t+h} - L_t| \geq \varepsilon)$ i.e. discontinuity occurs at random times.
(5) The paths of L_t are $\mathbb{P}$ almost surely right continuous with left limits (Cadlag paths which is defined in the theorem below).

Definition 5. *[8] Assume that $(\Omega, F, (F_t), \mathbb{P})$ is a filtered probability space and that L is a Wiener process in $\mathbb{R}^d$ adapted to (F_t). Then L is a Wiener process with respect to (F_t) or an (F_t)-Wiener process if, $\forall t, h \geq 0$, $L(t+h) - L(t)$ is independent of F_t.*

Definition 6. *[8] Let $q > 0$. A real-valued mean-zero Gaussian process $L = (L(t), t \geq 0)$ with continuous trajectories and covariance function*

$$\mathbb{E}L(t)L(s) = (t \wedge s)q, t, s \geq 0 \tag{19}$$

is called a Wiener process with diffusion q. If the diffusion is equal to 1 i.e. if $q = 1$ and

$$\mathbb{E}L(t)L(s) = (t \wedge s), t, s \geq 0 \tag{20}$$

then L is called ***standard Wiener process or standard Brownian motion****.*

Theorem 1. *Let $\{L(t)\}$ be a Lévy process. Then L_t has a Cadlag version which is also a Lévy process.*

Suppose we assuming that Theorem (1) is consistent for all Lévy process, then it is safe to take the Lévy process for this study as cadlag so that the **jump** of the Lévy process L_t at $t \geq 0$ is defined by

$$\Delta L_t = L_{t^+} - L_{t^-} \tag{21}$$

Taking L_{t^+} and L_{t^-} as positive and negative (upward and downward) movement in forwards curves respectively in an incomplete energy market. Therefore, Theorem 1 leads to the following assumption.

Theorem 2. *[2] Assume that $(L(t), t \geq 0)$ is a cadlag Lévy process in a Banach space B with jumps bounded by a fixed number $c > 0$; that is $|\Delta L(t)|_B \leq c$ for every $t \geq 0$. Then, for any $\beta > 0$ and $t \leq 0$,*

$$\mathbb{E}[e^{\beta |L(t)|_B}] < \infty \tag{22}$$

Proof: Adopting a similar consideration of a shorter proof in [9] we write $\tau_0 = 0$ and $\tau_{n+1} = \inf\{t \geq \tau_n : |L(t) - L(\tau_n)|_B \geq c\}$, $n = 0, 1, ...$ Since L has independent and stationary increments, the random variables $(\tau_{n+1} - \tau_n, n = 0, 1, ...)$ are independent and have the same distribution. Consequently, for $n = 1, 2, ...,$

$$\mathbb{E}[e^{-\tau_n}] = \mathbb{E}[e^{-(\tau_n - \tau_{n-1})...+(\tau_1 - \tau_0)}] = \prod_{j=1}^{n} \mathbb{E}e^{-(\tau_j - \tau_{j-1})} = \left(\mathbb{E}e^{-\tau_1}\right)^n = (\alpha)^n.$$

By Chebyshev's inequality,

$$\mathbb{P}\left(|L(t)|_B > 2nc\right) \leq \mathbb{P}\left(\tau_n < t\right) \leq e^t\left(\alpha\right)^n.$$

Note that $\alpha \in (0,1)$. Let $\gamma \in (0, \log 1/\alpha)$. Then

$$\mathbb{P}\left(exp\left\{\frac{\gamma}{2e}|L(t)|_B\right\} > e^{\gamma n}\right) \leq e^t\left(\alpha\right)^n,$$

and hence

$$\mathbb{E}exp\left\{\frac{\gamma}{2e}|L(t)|_B\right\} = \int_0^\infty \mathbb{P}\left(exp\left\{\frac{\gamma}{2e}|L(t)|_B\right\} > s\right) ds$$

$$\leq \sum_{n=0}^{\infty} \mathbb{P}\left(exp\left\{\frac{\gamma}{2e}|L(t)|_B\right\} > e^{\gamma n}\right) e^{\gamma(n+1)}$$

$$\leq e^{t+\gamma}\sum_{n=0}^{\infty}\left(\alpha e^{\gamma}\right)^n < \infty.$$

which gives the short proof to Theorem 2. Hence, in this study, we are concerned mainly with the case when E is a Hilbert Space $(U, \langle\cdot,\cdot\rangle_U)$ and ε is the σ-algebra of Borel sets $B(U)$. This processes further strengthens L_t, $\forall t \geq 0$ therefore, suppose that B_0 be the family of Borel sets $U \subset \mathbb{R}$ whose closure $\overline{U}$ does not contain 0. For $U \in B_0$, let $N(t,U)$ represent the number of jumps in the forwards curves described in a Lévy market of size $\Delta L_x \in U$ which occur before or at time t so that

$$N(t,U) = N(t,U,\omega) = \sum_{x:0<x\leq 1} \chi_U(\Delta L_x) \tag{23}$$

$N(t,U)$ is a Poisson random (jumps) measure for a forward curve in energy market in time $t_1, t_2 \geq 0$. It measures the sizes of the continuous sample path movement between two consecutive prices in the energy market. The differential form of this measure is written in the form of $N(dt,dz)$, and for times $0 \leq t_1 \leq t_2 < \infty$, we therefore denote the differential $N(dt,dz)$ as

$$N(dt,dz) = N(t_2,U) - N(t_1,U), 0 \leq t_1 \leq t_2 < \infty.$$

In (23) χ is generated measure while N is the number of jumps of electricity forwards prices in a Lévy market.

Remark 1. *Taking a finite $N(t,U)$ for all $U \in B_0$, and define*

$$T_1(\omega) = inf\{t > 0, L_t \in U\}$$

for all $T_1(\omega) > 0$ (almost surely). Then by right continuity of paths we have

$$\lim_{t \longrightarrow 0^+} L(t) = L(0) = 0, almost\ surely$$

Therefore, for all $\varepsilon > 0$ there exist $t(\varepsilon) > 0$ such that $|L(t)| < \varepsilon$ for all $t < t(\varepsilon)$. This implies that the Lévy process $L(t) \notin U$ for all $t < t(\varepsilon)$, if $\varepsilon < dist(0,U)$. By inductive definition

$$T_{n+1}(\omega) = inf\{t > T_n(\omega); \Delta L_t \in U\}.$$

Then by the above argument, $T_{n+1} > T_n$ (almost surely). Hence $T_n \longrightarrow \infty$ as $n \longrightarrow \infty$ (almost surely). Assume then that $T_n \to T \to \infty$. But then $\lim_{x \longrightarrow T^-} L(S)$ cannot exist. Thereby contradicting the existence of left limits of the paths. It is well-known that Brownian motion $\{B(t)\}_{t \geq 0}$ has stationary and independent increments. Thus $B(t)$ is also a Lévy process.

Theorem 3. *(1) The set function $U \longrightarrow N(t,U,\omega)$ defines a σ-finite measure on B_0 for each fixed t, ω.*

(2) The set function

$$v(U) = E[N(1,U)]$$

where $E = E_P$ denotes expectation with respect to P, also defines a σ-finite measure on B_0, called the Lévy measure of $\{L_t\}$.

(3) Fix $U \in B_0$. Then the process $\pi_U(t) = \pi_U(t,\omega) = N(t,U,\omega)$ is a Poisson process of intensity $\lambda = v(U)$.

2.5. Estimation of Jump Sizes. The size of jumps of the forward curves in the Poisson random measure is defined by some Lévy processes which upon decomposition helps to measure the impact size of each continuous sample path movement of the curve. Therefore, we start by introducing the Lévy process decomposition of the measure.

2.5.1. ***Lévy-Ito Decomposition.*** Let $L = (L(t), t \geq 0)$ be an U-valued Lévy process with characteristics $\left(\alpha, \sigma^2, v\right)$. The jump at time t is $\Delta L(t) = L(t) - L(t^-)$. Hence we obtain a Poisson random measure N on $\mathbb{R}^+ \times (U - 0)$, which has an intensity measure λv, by the definition;

$$N(t,A) = \{0 \leq s \leq t, \Delta L(t) \in A\},$$

for each $A \in B(U)$. The associated compensator is denoted by $\tilde{N}$, so

$$\tilde{N}(dt,dz) = N(dt,dz) - dtv(dz).$$

We say that $A \in B(U)$ is bounded below if $0 \notin \tilde{A}$ and $t \geq 0$ so that the compound Poisson process;

$$Y_k(t) = \sum_{0 \leq s \leq t} \Delta L(s) 1_{\{\Delta L(s) \in A\}} = \int_A zN(t,dz) \text{is finite a.s}$$

If A is bounded below and $A \subseteq B_\sigma(0)$ for some $\sigma > 0$, we may define

$$Z_k(t) = Y_k(t) - \int_A zv(dz) = \int_A zN(t,dz) - \int_A zv(dz)$$

$$Z_k(t) = \int_A z\tilde{N}(t,dz).$$

and according to [8], we write

$$Z(t) = \int_{B_1} z\tilde{N}(t,dz),$$

where B_1 is a Borel set containing $(A_n, n \in \mathbb{N})$ which itself is a sequence of Borel sets and $A_n^c = B_1 - A_n$ is bounded below. We then obtain the Lévy-Ito Decomposition as presented in Theorem (4);

Theorem 4. *[10] Let $\{L_t\}$ be a U-valued Lévy process with characteristics (α, σ^2, v), there exist a Brownian motion B with covariance σ^2 and an independent Poisson random measure N and $\mathbb{R}^+ \times (U - \{0\},)$, with intensity measure λv so that for each $t \geq 0$, the L_t decomposition is given by*

$$L_t = \alpha t + \sigma B(t) + \int_{|z|<1} z\tilde{N}(t,dz) + \int_{|z|\geq 1} zN(t,dz) \tag{24}$$

for some constants $\alpha, \beta \in \mathbb{R}$, $R \in [0, \infty]$.

Here the measure $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ is called the compensated Poisson random measure of $L(.)$ and $B(t)$ is an independent Brownian motion, $\nu(dz)$ is Lévy measure of the Lévy process respectively. Also, it is convenient to say that $\int_{|z|>1} zN(t,dz)$ is the sum of all jumps (finite many) of size bigger than one and $\int_{|z|<1} z\tilde{N}(t,dz)$ process is the sum of small jumps (of size smaller than 1). For any given constants $R > 0$ small jumps and big jumps can be defined as $|z| < R$ and $|z| \geq R$ respectively, such that the corresponding Lévy-Ito decomposition is given as

$$L_t = \alpha t + \sigma B(t) + \int_{|z|<R} z\tilde{N}(t,dz) + \int_{|z|\geq R} zN(t,dz) \tag{25}$$

Also, suppose $A \in \mathbb{B}_0$ the process

$$M_t = \tilde{N}(t,A) \text{is a martingale} \tag{26}$$

If $\alpha = 0$ and $R = \infty$, we call L_t a Lévy martingale. Note that for this study, we can always assign $R = 1$. In this study, we will be considering jumps of sizes greater than 1. Therefore, we have a measure

$$\tilde{N}(dt,dz) = \begin{cases} N(dt,dz) - \nu(dz)dt, if\, |z| < 1 \\ N(dt,dz), if\, |z| \geq 1 \end{cases} \tag{27}$$

For each $A \in B_0$ the process $M_t = \tilde{N}(t,A)$ is a martingale. Also, if $\alpha = 0$ and $R = \infty$, then L_t is a Lévy martingale.

Theorem 5. *Choosing $R = 1$ if*

$$E[L_t] < \infty, \forall t \geq 0,$$

then

$$\int_{|z|\geq 1} |z|\, \nu(dz) \leq \infty$$

and we may choose $R = \infty$ and hence write

$$L_t = \alpha_1 t + \beta B(t) + \int_{\mathbb{R}} z\tilde{N}(t,dz).$$

where

$$\alpha_1 = \alpha + \int_{|z|\geq 1} |z|\, \nu(dz).$$

which reduces Equation (25) because the jump size is greater than 1 at $\mathbb{R} = \infty$.

2.6. Finite and Infinite Activity. The rate at which such continuities arrive is given by the integral of the Lévy density, $\lambda = \int_{-\infty}^{+\infty} \upsilon(dx)$. In the particular case of compound Poisson processes, $\upsilon(dx) = \lambda f(x)dx$, where $f(x)$ is the density of the distribution of jumps. When this integral is finite the process is said to have finite activity. If the integral is infinite, then it is said to be infinite activity, which implies that the process can have infinite jumps in a finite time interval. So, a pure jump process has a triplet of the form $(a, 0, \upsilon)$, since there is no diffusion component. As for a pure diffusion process, the triplet is of the form $\left(a, \sigma^2, 0\right)$ since no jump component exists, whereas in a jump-diffusion process the triplet takes the form $\left(a, \sigma^2, \upsilon\right)$ for it is a mixture of jump and diffusion component. It is also important to point out that a jump-diffusion has a jump component with finite rate λ, whereas a pure jump process has, in financial modeling, infinite activity.

2.7. Lévy-Khintchine Representation. A fundamental theorem concerning Lévy processes is the so called Lévy-Khintchine representation:

Theorem 6. *Let $\{L_t\}$ be a Lévy process with Lévy measure v. Then*

$$\int_{\mathbb{R}} min(1, z^2)v(dz)$$

and its characteristic function is of the form

$$\phi L_t = E[e^{tu} L_t] = \int_{\mathbb{R}} e^{iuL_t} \upsilon(dx) = e^{t\psi(u)}, u \in \mathbb{R} \tag{28}$$

iff there exist a triplet $\left(a, \sigma^2, \upsilon\right)$, $a \in \mathbb{R}$, $\sigma \geq 0$, υ a measure concentrated on $\mathbb{R}\{0\}$ that satisfies $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \upsilon(dx) \leq \infty$ such that

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + i\alpha u + \int_{|z|<R} \left\{e^{iuz} - 1 - iuz\right\} v(dz) + \int_{|z|\geq R} (e^{iuz} - 1)vdz \tag{29}$$

where ψ is called the characteristics exponent.

Conversely, given constants α, σ^2 and a measure υ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} min(1, z^2)v(dz) < \infty.$$

There exist a Lévy process $L(t)$ (unique in law) such that (28) and (29) hold and it is possible that

$$\int_{|z|\geq 1} |z|\, v(dz) = \infty.$$

We also know that a Lévy process is a semimartingale. This theorem is the core of the theory of Lévy processes, and deserves much attention. It states that every Lévy process has a characteristic function of the form of equation (28). Thus, they can be parameterized using the triplet $\left(a, \sigma^2, \upsilon\right)$, where a is the drift, σ^2 is the variance of a Brownian motion and $\upsilon\,(dx)$ is the so called Lévy measure or jump measure.

Definition 7. *Let D_{ucp} denote the space of cadlag adapted processes, equipped with the topology of uniform convergence on compacts probability $(ucp) : H_n \longrightarrow H_{ucp}$ if for all*
$t > 0$ $sup_{0\leq s\leq t}\left|H_n(s) - H(s)\right| \longrightarrow 0$ in probability. Also, Let L_{ucp} denotes the space of adapted cadlaq process (left continuous with right limits) equipped with the ucp topology. If $H(t)$ is a step function of the from

$$H(t) = H_0\chi_{(0)}(t) + \sum_t H_i\chi_{T_i,T_{i+1}}(t).$$

where $H_t \in F_t$ and $0 = T_0 \leq T_1 \leq ... \leq T_{n+1} \leq \infty$ are F_t-stopping times and χ is cadlag, we define

$$J_\chi H(t) = \int_0^t H_s dL_s = H_0 L_0 + \sum_i H_i(L_{T_{i+1\wedge t}} - L_{T_{i\wedge t}}); t \geq 0.$$

Theorem 7. *Let L be a semimartingale. Then the mapping J_x can be extended to a continuous linear map*

$$J_x : L_{ucp} \longrightarrow D_{ucp}.$$

This constructions allows us to define stochastic integrals of the form

$$\int_0^t H(s) d\eta_s$$

for all $H \in Lucp$. In view of the decomposition (27) this integral can be split into integrals with respect to ds, $dB(s)$, $\tilde{N}(ds,dz)$ and $N(ds,dz)$. This makes it natural to consider the more general stochastic integral of the form

$$L(t) = L(0) + \int_0^t \alpha(s,\omega)ds + \int_0^s \sigma(s,\omega)dB(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,z,\omega)\tilde{N}(ds,dz) \tag{30}$$

where the integrads are $\mathbb{F}$-predictable are satisfying the appropriate growth conditions

$$\int_0^t \left\{ |\alpha(s)| + \sigma^2(s) + \int_{\mathbb{R}} \gamma^2(s,z)v(dz) \right\} ds < \infty, \forall t > 0$$

for the integrals to exist and for simplicity we put

$$\tilde{N}(dt,dz) = \begin{cases} N(dt,dz) - v(dz)dt, if\,|z| < R \\ N(dt,dz), if\,|z| \geq R \end{cases} \tag{31}$$

with R as in Theorem 3, then we use the following short hand differential notation for process $L(t)$ satisfying (30)

$$dL(t) = \alpha(t)dt + \sigma dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz) \tag{32}$$

where (32) is called the Ito-Lévy process and $B(t)$ is a Brownian motion with $\alpha(t)$, $\sigma(t)$ and $\gamma(t)$ satisfying the necessary growth conditions for which the given Ito-Lévy process has unique strong solution $L(t)$.

Since the semimartingale $M(t)$ is called a local martingale up to time T (with respect to P) then there exists an increasing sequence of F_t-stopping times τ_n such that $\lim_{n\longrightarrow\infty} \tau_n = T$.a.s and

$$M(t \wedge \tau_n), \text{is a martingale with respect to P for all n}$$

Such that

(1) If

$$E\left[\int_0^T \int_{\mathbb{R}} \gamma^2(t,z)v(dz)dt\right] < \infty \tag{33}$$

then the process

$$M(t) = \int_0^T \int_{\mathbb{R}} \gamma(s,z)\tilde{N}(ds,dz), 0 \leq t \leq T$$

is a martingale.

(2) If

$$\int_0^T \int_{\mathbb{R}} \gamma^2(t,z)v(dz)dt < \infty, \text{a.s.} \tag{34}$$

then $M(t)$ is a local martingale, $0 \leq t \leq T$.

The above conditions therefore leads to the important Ito-Lévy process if for $L(t)$ in Equation (32) and $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ is a C^2 function, is the process $Y(t) = f(t, L(t))$ again an Ito-Lévy process and let $L^{(c)}(t)$ be the continuous part of $L(t)$, i.e., $L^{(c)}(t)$ is obtained by removing the jumps from $L(t)$. Then an increment in $Y(t)$ stems from an increment in $X^{(c)}(t)$ plus the jumps in Equation (31). Therefore, based on the classical Ito formula we guess that;

$$dY(t) = \frac{\partial f}{\partial t}(t,L(t))dt + \frac{\partial f}{\partial x}(t,L(t))dX^{(c)}(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,L(t)) \cdot \sigma^2(t)dt \\ + \int_{\mathbb{R}} \left\{ f(t,L(t^+) + \gamma(t,z)) - f(t,L(t^-)) \right\} N(dt,dz).$$

which can support the guess since

$$dL^{(c)}(t) = \left(\alpha(t) - \int_{|z|<R} \gamma(t,z)v(dz) \right) dt + \sigma(t)dB(t),$$

which gives th following results; Suppose that $L(t) \in \mathbb{R}$ is an Ito-Lévy process of the form

$$dL(t) = \alpha(t,\omega)dt + \sigma(t,\omega)dB(t) + \int_{\mathbb{R}} \gamma(t,z,\omega)\tilde{N}(dt,dz), \tag{35}$$

where

$$\tilde{N}(dt,dz) = \begin{cases} N(dt,dz) - v(dz)dt, if\,|z| < R \\ N(dt,dz), if\,|z| \geq R \end{cases} \tag{36}$$

for some $R \in [0,\infty]$. Therefore, suppose $f \in C^2\left(\mathbb{R}^2\right)$ and define $Y(t) = f(t,L(t))$. Then $Y(t)$ is again an Ito-Lévy process and

(37)

$$dY(t) = \frac{\partial f}{\partial t}(t,L(t))dt + \frac{\partial f}{\partial x}(t,L(t))\left[\alpha(t,\omega)dt + \sigma(t,\omega)dB(t)\right] + \frac{1}{2}\sigma^2(t,\omega)\frac{\partial^2 f}{\partial x^2}(t,L(t))dt \\ + \int_{\mathbb{R}} \left\{ f(t,L(t^+) + \gamma(t,z,\omega)) - f(t,L(t^-)) \right\} - \frac{\partial f}{\partial t}(t,L(t^-))\gamma(t,z,\omega)v(dz)dt \\ + \int_{\mathbb{R}} \left\{ f(t,L(t^+) + \gamma(t,z)) - f(t,L(t^-)) \right\} \tilde{N}(dt,dz).$$

However, solving the stochastic differential equation (SDE) of a Geometric Lévy;

$$dL(t) = L(t^-)\left[\alpha dt + \sigma dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz) \right], \tag{38}$$

where α, σ are constants and $\gamma(t,z) \geq -1$. Therefore, in other to estimate $L(t)$ we rewrite Equation (38) as follows;

$$\frac{dL(t)}{L(t^+)} = \alpha dt + \sigma dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)$$

Now define

$$Y(t) = \ln L(t).$$

Then estimating by Ito's formula, we have;

$$\begin{aligned} dY(t) = \frac{dL(t)}{L(t)} &= [\alpha dt + \sigma dB(t)] - \frac{1}{2}\sigma^2 L^{-2}(t)L^2(t)dt \\ &+ \int_{|z|<R} \left\{\ln\left(L(t^-) + \gamma(t,z)L(t^-)\right) - \ln\left(L(t^-)\right) - L^{-1}(t^-)\gamma(t,z)L(t^-)\right\} v(dz)dt \\ &+ \int_{\mathbb{R}} \left\{\ln\left(L(t^-) + \gamma(t,z)L(t^-)\right) - \ln\left(L(t^-)\right)\right\} \tilde{N}(dz)dt \end{aligned} \tag{39}$$

$$\begin{aligned} dY(t) = \left(\alpha - \frac{1}{2}\sigma^2\right)dt + \sigma dB(t) + \int_0^t \int_{|z|<R} \{\ln(1+\gamma(s,z)) - \gamma(s,z)\} v(dz)ds \\ + \int_{\mathbb{R}} \ln(1+\gamma(s,z))\tilde{N}(ds,dz) \end{aligned} \tag{40}$$

Hence

$$\begin{aligned} Y(t) = Y(0) + \left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma dB(t) \\ + \int_0^t \int_{|z|<R} \{\ln(1+\gamma(s,z)) - \gamma(s,z)\} v(dz)ds \\ + \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,z))\tilde{N}(ds,dz) \end{aligned} \tag{41}$$

However, we recall that

$$Y(t) = \ln(L(t))$$

therefore, this gives the results

$$L(t) = L(0)e^W \tag{42}$$

where

$$\begin{aligned} W = \left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma dB(t) + \int_0^t \int_{|z|<R} \{\ln(1+\gamma(s,z)) - \gamma(s,z)\} v(dz)ds \\ + \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,z))\tilde{N}(ds,dz) \end{aligned}$$

Therefore, Equation (42) is called the geometric Lévy process which is analogous with the diffusion case where $N=0$ and it is the expected model for estimating the stock prices with the presence of jumps in the forward curve of energy market.

2.8. Estimating the Jump Parameters. The concept of a jump-diffusion process is that the diffusion part takes into consideration the normal fluctuation in the risk asset's price caused by a temporary imbalance in the supply and demand, changes in capitalization rates, changes in the economic outlook or other information that causes marginal changes in price of electricity in the energy markets. However, for the non-marginal variations, it is expected that the information that causes them come in discrete points in time, and that is why a jump component is added to the traditional diffusion process [11].

2.9. Nonparametric Estimation of Lévy Processes. According to the study by [12], estimation of Lévy process where the parameter σ in Equation (42) is consistently estimated based on data virtually observed in continuum. However, in this study, we consider the cases when the processes are observed at direct time points only. Suppose we have a discrete record of equidistant observations of a Lévy process $L(t)$, denoted by $\{L_0, L_\Delta, L_{2\Delta}, ..., L_{n\Delta}\}$, where Δ denotes the data frequency. Let the time horizon be $T = n\Delta$. The increments are given by

$$X_j = L_{j\Delta} - L_{(j-1)\Delta}, j = 1, 2, ..., n.$$

Therefore, in order to estimate the nonparameters with the frequency data in a high frequency framework, we propose the following method;

2.10. Threshold Method. A threshold is ideally defined for the excitement experienced in the forward prices in the energy market and whenever a movement is larger than the threshold, it is classified as it jump. In [11], a generalized jump-diffusion model of the form

$$dL_t = \alpha(t,\omega)dt + \sigma(t,\omega)dW_t + \gamma(t,\omega)dN_t, t > 0 \tag{43}$$

was considered where $|\alpha(t,\omega)| \leq M$, $|\sigma(t,\omega)| \leq M$ for some $M > 0$, and $\gamma(t,\omega) \geq c$ for some $c > 0$. Which implies that the diffusion coefficients are bounded from above, and the jump coefficient is bounded away from zero.

Further more, the contribution of the diffusion part to the increments $X_j, j = 1,2,...,n$ tends to approach zero immediately as Δ decreases, because the stochastic integral $\int \sigma(t,\omega)dW_t$, after a change of time, behaves as a Brownian motion. The Lévy modulus of continuity theorem for the path of a Brownian motion asserts that the rate $\sqrt{2\Delta\log(1/\Delta)}$ measures the speed at which the increment of a Brownian motion over time step Δ goes to zero. Therefore, suppose we propose that

$$r(\Delta) = \sqrt{16M\beta} \cdot \sqrt{\Delta\log(1/\Delta)}$$

for some $\beta \in (1,2]$ or simply put that $r(\Delta)$ is any other function of Δ which goes to zero more slowly, and conclude that a jump occurs if $\left|X_j > r(\Delta)\right|$.

However, consistent estimators of $\left\{N_{j\Delta}, j = 1,2,...n\right\}$ as constructed by [11], of the jump intensity λ and of the size of jump γ_{τ_j}, where τ_j are the instants of jump within the time interval $[0,T]$. The estimator of λ is asymptotically Gaussian. These results holds when the time horizon $T \longrightarrow \infty$, the data frequency $\Delta \longrightarrow 0$, such that $n\Delta^\beta \longrightarrow 0$ for $\beta \in (1,2]$.

Similarly, the threshold method was also discussed in [13], where a general Lévy process was considered. [13] constructed consistent estimators of the diffusion parameters and of the Lévy measures for both the finite-activity case and the infinite-case. In the finite case, the estimators are seen to be asymptotically normally distributed. However, this results are true only when $T \longrightarrow \infty$ and $\Delta \longrightarrow 0$ such that $T\Delta \longrightarrow 0$.

2.11. Testing for Jumps. The previous sections have been explicit about the sizes and directions of jumps and jump diffusion in Lévy processes. Therefore, after obtaining the estimates for all the components of the finite-activity Lévy process, it is of grave importance that the classification problem is considered, given an increment of the process which is classified as a jump or non-jump as the case may be. This problem is related to the testing of jumps. Although various literature have proposed different methods of testing for jumps in a Lévy process, a non-parametric statistics which tests whether a jump has occurred or not was however proposed by [14] and given by:

$$S_n = \frac{\sum_{i=1}^{[n/K]} \left|Y_{iK\Delta} - Y_{(i-1)K\Delta}\right|^p}{\sum_{i=1}^{n} \left|Y_{i\Delta} - Y_{(i-1)\Delta}\right|^p} \tag{44}$$

where S_n is the preliminary estimator, $Y_i K\Delta$, $Y_{i-1}K\Delta$ Lévy processes with data frequency Δ, n is the number of observations, while K is a bounded function, p is probability measure. Therefore, Equation (1) converges to two different deterministic constants depending on whether the process has jumps or not. Although [14] discussed the testing of the jumps only, however, it was discovered that the test is valid for all Ito semi-martingales. In this study, however, the Bayes procedure according to [11] was used to classify the observations with or without jump which shows theoretically that the data frequency $\Delta > 0$ whenever the misclassification probability $P_n \longrightarrow 0$.

2.12. Estimation of the Jump Frequency with Known Diffusion Parameters. In this section, we consider a proposed method which estimates the jump components of the jump-diffusion models with the presence of jump frequency and the jump size distribution where the frequency of the observed data is fixed and the parameters in the diffusion components (drift and volatility) are known. This proposed method is based on a characteristics function approach for a discretely observed realization of jump-diffusion model with the length of the time interval fixed. Then each increment may or may not involve a jump (or jumps). If there is no jump, then the model is simply an increment of the diffusion component. Since on average the jump size is typically much larger than the increment of the diffusion component, the increment of the mode with a jump (or jumps) would correspond to a distribution with much heavier tails than the increment of the model without a jump does. Also, while using the characteristics function techniques, the distribution is transformed with heavier tails to a characteristics function converging to zero always at a faster rate. By capturing this difference in the convergence rates of the characteristics functions, we construct a new estimator. However, the approach of estimating a jump frequency with known parameters has the following features;

- The sample size n increases by increasing the time horizon τ_n (i.e. the total number of years) of the data set while keeping the data frequency Δ fixed.
- The distribution of the sample remains the same.
- Since the data frequency Δ is fixed, the jump frequency can be explained by the *jump ratio*, i.e. the expected percentage of jump observations among all observations.

2.13. Proposed Model Settings. Let $\{L_\Delta, L_{2\Delta}, ..., L_{n\Delta}\}$ be a discretely observed realization with a finite activity, where Δ is the length of the time interval between two consecutive observations, and $T = n\Delta$ is the time horizon, from the following jump-diffusion model in Equation (**??**) which represent the log-price return, $\ln S_t$ of energy market derivatives:

$$L_t \equiv \ln S_t = L_0 + \mu_0 t + \sigma_0 W_t + \sum_{k=1}^{N_t} Y_k \tag{45}$$

with $\mu_0 = \left(\alpha - \frac{\sigma^2}{2} - \lambda\zeta\right)$ as defined in Equation (**??**) and to satisfy the martingale property we set $L_0 = 0$ so that Equation (45) reduces to;

$$L_t \equiv \ln S_t = \mu_0 t + \sigma_0 W_t + \sum_{k=1}^{N_t} Y_k \tag{46}$$

where $\{W_t\}$ is the standard Brownian motion, $\{N_t\}$ is a Poisson process with jump intensity parameter λ, $Y_1, Y_2, ...,$ are independently and identically distributed (i.i.d.) random variables representing jump sizes, and (W_t, N_t, Y_k) are mutually independent. This process L_t consists of two components" the first part is the diffusion component $\mu_0 t + \sigma_0 W_t$ and the second part is the jump component $\sum_{k=1}^{N_t} Y_k$.

For the purpose of this study and the result of Equation (46) we state the following assumptions concerning the model setting;

- **A1** : The distribution of the jump size Y is absolutely continuous with respect to the Lebesgue measure.
- **A2** : The product $\lambda\Delta$ is small so that the term of order $\o(\lambda\Delta)$ in the expansion $e^{-\lambda\Delta} = 1 - \lambda\Delta + o(\lambda\Delta)$ is negligible.
- **A3** : μ_0 and σ_0 are known in order to separate the error due to the estimation of the diffusion parameters from the error due to the estimation of the jump component.

However, the assumption presented in (**A1**) is equivalent to the one that the distribution of jump size Y has a density function with respect to the Lebesgue measure. Meanwhile, in order to justify assumption (**A2**), we consider the following example: when $\Delta = 1/365$(daily data in the year) and $\lambda = 11.25$, then $\lambda\Delta = 3 percent$, which gives us $e^{-\lambda\Delta} = 0.9696$ and $1 - \lambda\Delta = 0.9692$.

In [11], $\lambda = 12.5$ and in [15], $\lambda = 10$. Here we use $\lambda = 11.25$ as average of both.

The difference between them is 0.0004 which is very small and as a result the assumption (**A2**) seems to be reasonable.

2.14. Methodology for Testing Jumps. The estimators of jump frequency and jump size distribution that we propose are based on characteristics functions. The *characteristic function* (c.f.) of a random variable Y is defined by

$$\varphi_Y = \mathbb{E}\left[e^{itY}\right], t \in$$

Its real and imaginary parts are denoted by

$$\Re\left(\varphi_Y(t)\right) = \mathbb{E}[cos(tY)] = \int_{\mathbb{R}} cos(ty) \cdot f_Y(y)dy$$

and

$$\Im\left(\varphi_Y(t)\right) = \mathbb{E}[sin(tY)] = \int_{\mathbb{R}} sin(ty) \cdot f_Y(y)dy,$$

respectively, where f_Y is the pdf of the random variable Y The following lemma describes the tail behaviour of the characteristic function of an absolutely random variable.

Lemma 1. *When a random variable is absolutely continuous, its characteristic function satisfies*

$$\lim_{|x| \longrightarrow \infty} \varphi(t) \longrightarrow 0.$$

With the above Lemma we will find the characteristic function of increments of process L_t in Equation (46). Let us denote the increments of L_t by

$$X_j = L_{j\Delta} - L_{(j-1)\Delta} \tag{47}$$

$$= \mu_0\Delta + \sigma_0\left(W_{j\Delta} - W_{(j-1)\Delta}\right) + \sum_{k=N_{(j-1)\Delta}+1}^{N_{j\Delta}} Y_k, j = 1,2,...,n \tag{48}$$

Then the variables $\left\{X_j, j = 1,2,...,n\right\}$ are i.i.d. Denote the increments of the diffusion component by

$$Z_j \equiv \mu_0\Delta + \sigma_0\left(W_{j\Delta} - W_{j-1}\Delta\right), j = 1,2,...,n \tag{49}$$

Then $\left\{Z_j, j = 1,2,...,n\right\}$ are i.i.d. and $Z_j \sim N\left(\mu_0\Delta, \sigma_0^2\Delta\right)$. Let X be a random variable that has the same distribution as the increments X_j's, and Z be a random variable withthe smae distribution as Z_j's. Since $X_j = Z_j + \sum_{K=N_{(j-1)\Delta}+1}^{N_{j\Delta}} Y_k$, the characteristic function of X is given by [

$$\varphi_Y(t) = \mathbb{E}\left[\exp\left\{itX_j\right\}\right] \tag{50}$$

$$= \mathbb{E}\left[\exp\left\{it\left(Z_j + \sum_{k=N_{(j-1)\Delta}+1}^{N_{j\Delta}} Y_k\right)\right\}\right] \tag{51}$$

$$= \mathbb{E}\left[\exp\left\{it\left(Z_j + \sum_{k=1}^{N_\Delta} Y_k\right)\right\}\right] \tag{52}$$

$$= \varphi_Z(t) \cdot \mathbb{E}\left[e^{it\sum_{k=1}^{N_\Delta} Y_k}\right] \tag{53}$$

where $\varphi_Z(t) = e^{it\mu_0\Delta - t^2\sigma_0^2\Delta/2}$ is the characteristic function of Z, the third line is due to the fact that $\sum_{k=N_{(j-1)\Delta}+1}^{N_{j\Delta}} Y_k$ and $\sum_{k=1}^{N_\Delta} Y_k$ have the same distribution, and in the last line the independence between Z_j and (N_t, Y) is used. By conditioning on the number of jumps occurring in the time interval $[0,\Delta]$, we have

$$\mathbb{E}\left[e^{it\sum_{k=1}^{N_\Delta} Y_k}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{it\sum_{k=1}^{N_\Delta} Y_k}\middle| N_\Delta\right]\right] \tag{54}$$

$$= \sum_{l=0}^{\infty} \mathbb{E}\left(e^{it\sum_{k=1}^{N_\Delta} Y_k}\right) \cdot P\left(N_\Delta = l\right) \tag{55}$$

$$= \sum_{l=0}^{\infty} \left(\varphi_Y(t)\right)^l \cdot \frac{(\lambda\Delta)^l}{l!} e^{-\lambda\Delta} \tag{56}$$

$$= e^{-\lambda\Delta(1-\varphi_Y(t))} \tag{57}$$

Therefore, by Equation (53), we obtain the characteristic function of X as

$$\varphi_X(t) = \varphi_Z(t) \cdot e^{-\lambda\Delta(1-\varphi_Y(t))} \tag{58}$$

Due to the assumption (**A2**) and the fact that $\|1 - \varphi_Y(t)\| \leq 2$, we have

$$e^{-\lambda\Delta(1-\varphi_Y(t))} = 1 - \lambda\Delta\left(1 - \varphi_Y\right) + o\left(\lambda\Delta\right),$$

and the error term $o\left(\lambda\Delta\right)$ is negligible.

Let us denote

$$\alpha \equiv \alpha(\Delta) = \lambda\Delta. \tag{59}$$

Note that

$$\mathbb{P}(N_\Delta \geq 1) = 1 - \mathbb{P}(N_\Delta = 0) = 1 - e^{-\lambda\Delta} = 1 - (1 - \lambda\Delta + o(\lambda\Delta)) = \lambda\Delta + o(\lambda\Delta),$$

that is, under the assumption (**A2**), σ provides a good approximation of the probability of at least one jump occurring during a time step of length Δ. Thus, we may call α the *jump ratio*, since it approximate the expected proportion of increments with jumps. Now, if we ignore the term $o(\lambda\Delta)$, we can write Equation (58) as

$$\varphi_X(t) = \varphi_Z \cdot (1 - \alpha + \alpha\varphi_Y(t)) \tag{60}$$

Representation of Equation (60 is an important formula, which we will use in the following to derive estimates of the jump frequency and the jump size distribution. Dividing both sizes of (60) by $\varphi_Z(t)$, we get

$$\frac{\varphi_X(t)}{\varphi_Z} = 1 - \alpha + \alpha\varphi_Y(t) \tag{61}$$

By assumption (**A1**) and Lemma (1), we have

$$\lim_{t \longrightarrow \infty} \varphi_Y = 0.$$

Then, by taking limits of both sides of Equation (61), we get

$$\alpha = 1 - \lim_{t \longrightarrow \infty} \frac{\varphi_X(t)}{\varphi_Z(t)}. \tag{62}$$

Since α is a real number, we may take the real part of the right-hand side to obtain

$$\alpha = 1 - \lim_{t \longrightarrow \infty} \Re\left(\frac{\varphi_X(t)}{\varphi_Z(t)}\right). \tag{63}$$

To characterize the distribution of Y, we use equation (60) again to obtain

$$\varphi_Y = \frac{\frac{\varphi_X}{\varphi_Z} - (1 - \alpha)}{\alpha} \tag{64}$$

The two equations (63) and (64) form the basis to estimate the jump frequency and the jump size distribution.

2.15. Proposed Estimator. Since Z follows normal distribution $Z \sim N\left(\mu_0\Delta, \sigma_0^2\Delta\right)$, we may define for a given data frequency Δ,

$$\mu = \mu_0\Delta, \sigma = \sigma_0\sqrt{\Delta} \tag{65}$$

Then the characteristic function of Z is given by

$$\varphi_Z(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2} \tag{66}$$

By assumption (**A3**), μ and σ are known. Thus using Equation (63) we may define an estimator of the jump ratio α by

$$\tilde{\alpha} = 1 - \lim_{t \longrightarrow \infty} \Re\left(\frac{\tilde{\varphi}_X(t)}{e^{i\mu t - \frac{1}{2}\sigma^2 t^2}}\right) \tag{67}$$

where $\tilde{\varphi}_X$ is an estimator of the characteristic function of X. After obtaining $\tilde{\alpha}$, we can use Equation (64) to introduce an estimator of the distribution of the jump size Y in terms of its characteristics function by;

$$\tilde{\varphi}_Y(t) = \frac{\frac{\tilde{\varphi}_X(t)}{e^{i\mu t - \frac{1}{2}\sigma^2 t^2}} - (1 - \tilde{\alpha})}{\tilde{\alpha}} \tag{68}$$

Therefore, in order to define an estimator completely, we have to address the following issues;

- how to obtain an estimator of $\tilde{\varphi}_X(t)$, where X is observable.
- how to deal with the limit of t going to infinity in Equation (67). An appropriate selection of t might be necessary.

Therefore, we start with the method of estimating the c.f. of X by using the empirical characteristics function.

Suppose we have a random sample $X_1, X_2, ..., X_n$ from the distribution of X, the empirical characteristic function (e.c.f.) of X is defined by

$$\tilde{\varphi}_X(t) = \frac{1}{n}\sum_{j=1}^{n} e^{itX_j}, t \in \mathbb{R} \tag{69}$$

However, we introduce the strong law of large numbers which shows clearly, though empirically that the e.c.f. is strongly consistent estimator of the underlying characteristic function, i.e. $\tilde{\varphi}_X(t) = \varphi_X(t)$ a.s. for any fixed $t \in \mathbb{R}$. Therefore, it can be conveniently stated that the estimator of the jump frequency and the estimator of the jump distribution, the ratio of characteristic function $\varphi_X(t)/\varphi_Z(t)$ plays very significant role.

Now suppose we $\varphi_Y(t) = \varphi_V(t)$ and using the e.c.f. (69) together with the c.f. of Z, we estimate the c.f. of V by;

$$\tilde{\varphi}_V(t) = \varphi_X(t)/\varphi_Z(t) \tag{70}$$

$$= \frac{1}{n}\sum_{j=1}^{n} e^{itX_j} \cdot e^{-i\mu t+\frac{1}{2}\sigma^2 t^2} \tag{71}$$

$$= e^{\frac{1}{2}\sigma^2 t^2}\frac{1}{n}\sum_{j=1}^{n} e^{i\left(X_j-\mu\right)t} \tag{72}$$

$$= e^{\frac{1}{2}\sigma^2 t^2}\left\{\frac{1}{n}\sum_{j=1}^{n} cos\left(X_t-\mu\right)t + i\cdot\frac{1}{n}\sum_{j=1}^{n} sin\left(X_j-\mu\right)t\right\}. \tag{73}$$

Thus, the estimator (67) becomes;

$$\tilde{\alpha} = 1 - \lim_{t\longrightarrow\infty}\Re\left(\tilde{\varphi}_V(t)\right) \tag{74}$$

$$= 1 - \lim_{t\longrightarrow\infty}\left\{e^{\frac{1}{2}\sigma^2 t^2}\cdot\frac{1}{n}\sum_{j=1}^{n} cos\left(X_j-\mu\right)t\right\} \tag{75}$$

However, because in practice the use of a finite t has more significant results, Equation (74) involving the limit of t going to infinity with the limit that doesn't seem to exist is not valid for this study and hence its estimator cannot be used directly. So we consider an estimator of the form;

$$\tilde{\alpha} = 1 - \Re\left(\tilde{\varphi}_V(t)\right) \tag{76}$$

$$= 1 - e^{\frac{1}{2}\sigma^2 t^2}\cdot\frac{1}{n}\sum_{j=1}^{n} cos\left(X_j-\mu\right)t \tag{77}$$

with $t > 0$, and then with an appropriately selected value of t. Since α is allowed to take values only in the interval $[0,1]$, we may trim $\tilde{\alpha}(t)$ by one from above and zero from below.

3. Main Results

Electricity prices exhibit jumps in prices at periods of high demand when additional, less efficient electricity generation methods are brought on-line to provide a sufficient supply of electricity. In addition, they have a prominent seasonal component, along with reversion to mean levels.

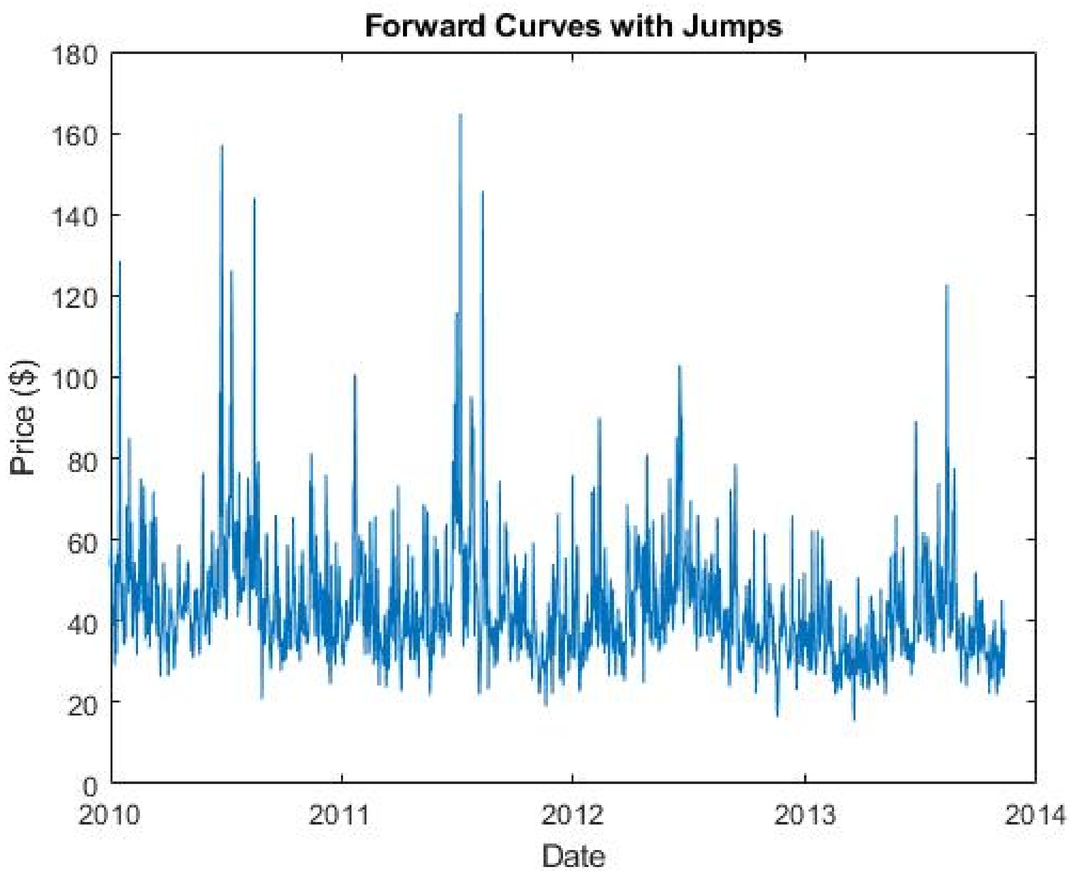

FIGURE 1. Forward Curve with Jumps (Data Source: Sample electricity prices from January 1, 2010 to November 11, 2013)

Sample electricity prices from January 1, 2010 to November 11, 2013 are loaded and plotted below. The fluctuation in prices of electricity in Figure 1 demonstrates on the forward curve the direction and sizes of jumps the energy market experience. Without further curve fittings and normalization, the curve shows uniform spikes between mid-year 2010 and early 2011 as well as end-year 2011 and early 2012, with a lesser spike in price between 2012 and 2013 as well as end-year 2013 and early 2014. The volatility causing jumps in the Lévy market calls for bench marking of the prices in the electricity market and hence suggest that for a proper fitting model to cushion for the losses there is need to set a threshold as defined in 43 with the diffusion component $|\alpha(t,\omega)| \leq M$, being bounded from above.

The deterministic seasonality part is calibrated using the least squares method. After the calibration the seasonality is removed from the logarithm of price. The logarithm of price and seasonality trends and the de-seasonalized logarithm of price are presented in Figure 2 below.

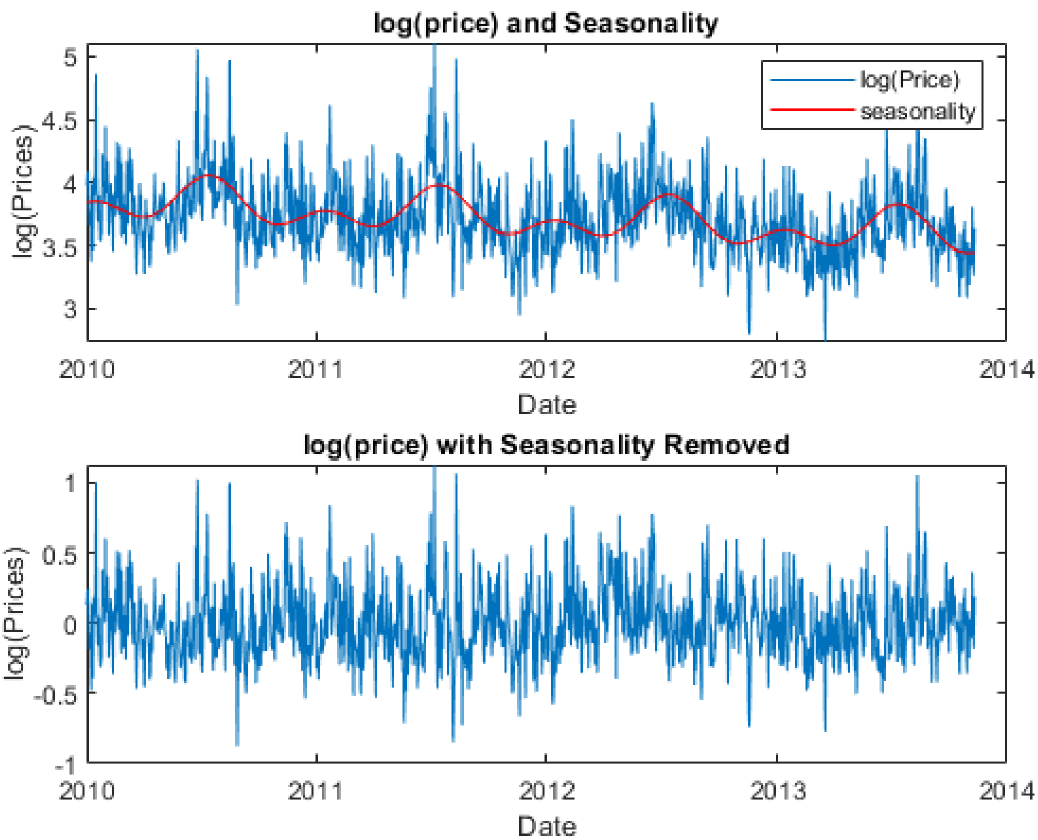

FIGURE 2. Calibration of parameters for the seasonality model (Data Source: Sample electricity prices from January 1, 2010 to November 11, 2013)

Finally the calibrated parameters and the discretized model allow us to simulate electricity prices under the real-world probability. The Monte Carlo simulation is conducted for approximately 2 years with 10,000 trials.It exceeds 2 years to include all the dates in the last month of simulation. This is because the expected simulation prices for the forwards contract expiry date is relevant to estimating the market price risk. The seasonality is added back on the simulated paths and a plot for a single simulation path is plotted in 3 below.

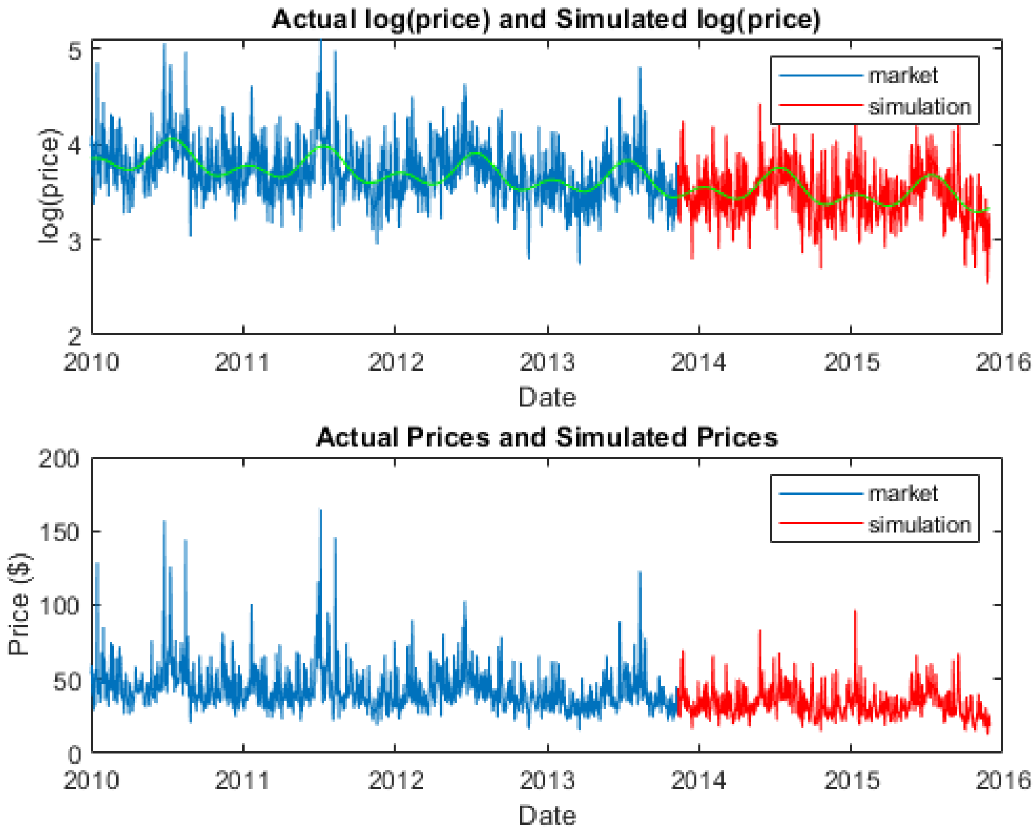

FIGURE 3. Monte Carlo Simulation of electricity data (Data Source: Sample electricity prices from January 1, 2010 to November 11, 2013)

3.1. Conclusions. Evaluating the relationship between the forward prices and the term structure helps energy stock marketers understand the direction of movement of energy prices. However, with the high spikes and volatility due to the incompleteness of the market, proper focus is placed on the jump processes which is defined by the Lévy stochastic nature of the market. In this study we understand that the relationship between the forwards and futures contract is close hence we adopt the two terminology as one due to the fact that the forward curves bench markets a more flexible payout deal for investors willing to take advantage of positive spikes in the prices of energy forwards in the contract.

Also, in adopting the Musiela parametrization as introduced by [3] gives a proper representation of the forward rates in the different time as an OTC market characteristics and this enables the definition of the forwards curve as a SPDE. In practice, the SPDE formulated with only

Brownian motion or the Poisson process may be ineffective in describing the complex dynamics such as estimating jumps in forward curves, hence the reason behind the discritization of sample paths and the evaluation of the price volatility in a filtered probability space and this enables the capturing of the bench mark of the energy prices in the forwards curve (term structure). It is recommended therefore, that investors in the energy market use a stop loss strategy based on the benchmark provided for the forward curve in the Lévy market in order to forestall loses and enhance profit margins.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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