Available online at http://scik.org
Math. Finance Lett. 2021, 2021:2
https://doi.org/10.28919/mfl/5851
ISSN: 2051-2929

A SUBDIFFUSIVE LÉVY MODEL FOR PRICING POWER OPTIONS IN ILLIQUID MARKETS

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Abstract. We extend the existing subdiffusive jump-diffusion model to a more general subdiffusive Lévy model, where the underlying Lévy process is time changed by a general inverse Lévy subordinator. We are able to obtain the characteristic function of log asset price by exploiting the fact that the Laplace transform of the inverse subordinator can be computed through an inverse Laplace transform in general and is given in explicit form for commonly encountered inverse subordinators. Different from previous studies where numerical methods such as Monte Carlo or PDE are used to calculate the option prices, we employ Fourier transform to derive the analytical solutions to power option prices.

Keywords: subdiffusion; Lévy process; Lévy subordinator; Fourier transform; power option.

2010 AMS Subject Classification: 91G20, 91G30, 91G80.

1. Introduction

Some financial data, especially asset prices can remain unchanged for constant periods. This feature is most common in illiquid markets where the number of participants, and thus the

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Received May 28, 2021

number of transactions, is rather low. The similar behavior is also observed in physical systems with subdiffusion, where the constant periods correspond to the trapping events in which the subdiffusive particles get immobilized. To describe and model the subdiffusion, the most common approach is to combine two independent processes, where the first process is some Markov process $X(t)$ and the second is an inverse subordinator $T(t)$ of a general infinite divisible distribution. Through the composition of the two processes, the new process, $X(T(t))$, can exhibit constant time periods and the length of constant periods follows the probability law of the specified Lévy subordinator.

In the subdiffusion literature for finance, $X(t)$ is typically modeled as a continuous diffusion process. Some important examples include geometric Brownian motion (GBM) in [9], Brownian motion in [10], multidimensional GBM in [7], Ornstein Uhlenbeck (OU) process in [3] and [5], Cox, Ingersoll and Ross (CIR) and 3/2 processes in [14] and constant elasticity of variance (CEV) process in [15]. Recently, [4] extend the existing literature by allowing $X(t)$ to be a jump diffusion so that the asset price can exhibit discontinuities caused by jumps.

It is well-known that it is a challenging task to obtain analytical formulas for option prices under subdiffusion. We can express the formula of option prices in terms of the integral of the density of the inverse subordinator. However, the density function can be expressed in closed form only in some special situations. Even if the density can be evaluated in closed form, numerical integration is needed to calculate the option prices. Instead, numerical methods such as Monte Carlo simulation ([7], [9], [10]) and PDE ([4], [8]) are often used to compute option prices. In [14] and [15], the analytical formulas for option prices are expressed in terms of eigenfunction expansion, which avoids the evaluation of density function and also eliminates the need for numerical integration. However, this method is feasible only when the underlying process $X(t)$ belongs to a specific class of diffusions.

The contributions of this paper are two-fold. Firstly, we extend the subdiffusive jump diffusion model of [4] to a more general subdiffusive Lévy model where the underlying process $X(t)$ can not only be Brownian motion or jump diffusion, but also any other Lévy processes such as Normal Inverse Gaussian (NIG), Variance Gamma (VG) and Carr, Geman, Madan and Yor (CGMY), to name a few; see [12] and references therein. At the same time, there is no any restriction imposed on the range of processes permissible for inverse subordinator $T(t)$. Some examples of processes for $T(t)$ that have been studied previously include inverse α-stable subordinator, inverse transient subordinator, inverse tempered α-stable subordinator (see e.g. [3]) and inverse Poisson subordinator (see e.g. [4]).

Secondly, we demonstrate how to price power options under subdiffusive Lévy processes and derive the closed-form solutions to option prices. The power option belongs to exotic options where the payoff depends on a power function of asset price. When the power parameter is one, the power option reduces to a vanilla option. The technique that enables us to obtain the analytical formulas for the option prices is Fourier transform. We exploit an important result that the Laplace transform of Laplace transform of $T(t)$ is given in explicit form. Therefore, the Laplace transform of $T(t)$ can be obtained by taking inverse Laplace transform in general. For most of inverse subordinators studied in the literature, their Laplace transform can be expressed in an explicit form. We are then able to derive the characteristic function of log asset price and calculate the option prices by inverting the characteristic function numerically. Other more efficient characteristic-function based methods such as fast Fourier transform (FFT) of [1] and Fourier cosine method of [2] can also be employed to obtain the option prices.

In a numerical study, we investigate the behaviour of the newly proposed model by comparing the subdiffusive Lévy model with the Lévy model. We demonstrate that the subdiffusive model can capture the constant periods observed in illiquid markets and different inverse subordinators can replicate different types of liquidity. Furthermore, we also illustrate that the volatility skew flattens out more slowly under subdiffusive Lévy model, which indicates it has the potential to capture the term structure of implied volatility better than the Lévy model.

The structure of the paper is as follows. Section 2 introduces the subdiffusive model. Several examples of Lévy processes are given and the Laplace transform of the inverse Lévy subordinator is provided with some examples. Section 3 derives the characteristic function for log asset price and the analytical formulas for the power option prices through Fourier transform. Section 4 numerically explores the behavior of some specific subdiffusive models with different underlying Lévy processes and inverse subordinators.

2. A SUBDIFFUSIVE LÉVY MODEL

Let $(\Omega, \mathscr{F}, P)$ be a probability space with an information filtration $(\mathscr{F}_t)$. Suppose under the physical probability measure P, the dynamics of an asset price $S(t)$ is given by

$$S(t) = S(0)\exp[Y(t)]\ , \tag{2.1}$$

where $Y(t)$ is a composition of two processes:

$$Y(t) = L(T(t))\ , \tag{2.2}$$

where $L(t)$ is a Lévy process and $T(t)$ is an inverse subordinator.

A Lévy process is infinitely divisible and its characteristic function can be expressed using Lévy-Khintchine formula (see e.g. [11]).

Proposition 2.1. *The characteristic function of the Lévy process $L(t)$ has the form*

$$\Phi_{L(t)}(u) = \mathrm{E}[\exp(\mathrm{iuL(t)})] = \exp(-\mathrm{t}\psi_{\mathrm{L}}(-\mathrm{iu})), \tag{2.3}$$

where $\psi_L(u)$ is the characteristic exponent of the Lévy process and given by

$$\psi_L(u) = \mu u - \frac{1}{2}\sigma^2 u^2 + \int_{(-\infty,\infty)} \left(1 - \exp(-ux) - ux1_{|x|<1}\right) \nu(dx), \tag{2.4}$$

where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$ and ν is a positive Radon measure on $\mathbb{R} \setminus \{0\}$ satisfying:

$$\int_{(-\infty,\infty)} (x \wedge 1)\nu(dx) < \infty.$$

We list several examples of Lévy processes and we refer to [12] for a more comprehensive list.

Example 2.2. Gaussian Process.

$$\psi_L(u) = \mu u - \frac{1}{2}\sigma^2 u^2. \tag{2.5}$$

Example 2.3. Jump Diffusion Process.

$$\psi_L(u) = \mu u - \frac{1}{2}\sigma^2 u^2 - \lambda\left[\exp(\Phi_X(iu)) - 1\right], \tag{2.6}$$

where $\lambda > 0$ is the intensity parameter for a Poisson process and $\Phi_X(\cdot)$ is the characteristic function of a jump size random variable.

Example 2.4. NIG Process.

$$\psi_L(u) = \mu u - \frac{1}{2}\sigma^2 u^2 + \delta\left(\sqrt{a^2-(b-u)^2} - \sqrt{a^2-b^2}\right), \tag{2.7}$$

where $a > 0$, $|b| < a$ and $\delta > 0$.

Example 2.5. CGMY Process.

$$\psi_L(u) = \mu u - \frac{1}{2}\sigma^2 u^2 - c\Gamma(-\gamma)\left[(M+u)^\gamma - M^\gamma + (G-u)^\gamma - G^\gamma\right], \tag{2.8}$$

where $\Gamma(\cdot)$ is the Gamma function and c, G, $M > 0$, $\gamma < 2$.

The process $T(t)$ in (2.2) is an inverse subordinator defined as

$$T(t) = \inf\{\tau > 0 : D(\tau) > t\}\ , \tag{2.9}$$

where $D(t)$ is a Lévy subordinator independent of $L(t)$.

The Lévy subordinator $D(t)$ is a nondecreasing process with positive jumps and non-negative drift with the Laplace transform given in the following result (see e.g. [11]).

Proposition 2.6. *The Laplace transform of $D(t)$ is given by*

$$\mathrm{E}[\exp(-\mathrm{u}\mathrm{D}(\mathrm{t}))] = \exp(-\mathrm{t}\psi_{\mathrm{D}}(\mathrm{u}))\ , \tag{2.10}$$

where ψ_D is the Lévy characteristic exponent and given by

$$\psi_D(u) = \gamma u + \int_{(0,\infty)} (1-\exp(-us))\nu(ds)\ , \tag{2.11}$$

where $\gamma \geq 0$ and the Lévy measure ν must satisfy

$$\int_{(0,\infty)} (s \wedge 1)\nu(ds) < \infty\ .$$

Throughout the whole paper we will always assume that ν additionally satisfies:

$$\int_{(0,\infty)} |x|^2 \nu(dx) < \infty.$$

Let $\eta(u,t)$ be the Laplace transform of the inverse subordinator $T(t)$, namely

$$\eta(u,t) = \mathrm{E}[\exp(-\mathrm{u}\mathrm{T}(\mathrm{t}))]\ . \tag{2.12}$$

To compute $\eta(u,t)$, we can first use the following result to calculate the Laplace transform $\hat{\eta}(u,k)$ of $\eta(u,t)$ (see e.g. [3], [14]).

Lemma 2.7. *The Laplace transform $\hat{\eta}(u,k)$ of $\eta(u,t)$ is*

$$\hat{\eta}(u,k) = \int_0^\infty \exp(-kt)\eta(u,t)dt = \frac{\psi_D(k)}{k(u+\psi_D(k))}\ , \tag{2.13}$$

where ψ_D is the Lévy exponent defined in (2.11).

Given the Laplace transform of $\eta(u,t)$, we can obtain $\eta(u,t)$ for general inverse subordinators by taking inverse Laplace transform. For commonly encountered inverse subordinators $T(t)$, we can obtain the explicit expression for $\eta(u,t)$ (see e.g. [3], [4] for detailed derivations).

Example 2.8. Inverse α-stable subordinator. Let $D(t)$ be a stable subordinator with Lévy exponent $\psi_D(k) = k^\alpha$ with $\alpha \in (0,1)$. Then, $\eta(u,t)$ for $T(t)$ is

$$\eta(u,t) = E^1_{\alpha,1}(-ut^\alpha)\ , \tag{2.14}$$

where $E^\gamma_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function:

$$E^\gamma_{\alpha,\beta}(z) = \sum_{j=0}^\infty \frac{(\gamma)_j z^j}{j!\Gamma(\alpha j+\beta)}\ ,$$

where α, β, $\gamma \in \mathbb{C}$, $\mathbb{Re}(\alpha)$, $\mathbb{Re}(\beta)$, $\mathbb{Re}(\gamma) > 0$.

Example 2.9. Inverse transient subordinator. Let $D(t)$ be a transient subordinator with Lévy exponent $\psi_D(k) = c_1k^{\alpha_1} + c_2k^{\alpha_2}$ with α_1, $\alpha_2 \in (0,1)$, $\alpha_1 < \alpha_2$, c_1, $c_2 \geq 0$ and $c_1 + c_2 = 1$. Then, we have

$$\begin{aligned}\eta(u,t) = &\sum_{j=0}^\infty \left(-\frac{c_1t^{\alpha_2-\alpha_1}}{c_2}\right)^j E^{j+1}_{\alpha_2,(\alpha_2-\alpha_1)j+1}\left(-\frac{ut^{\alpha_2}}{c_2}\right)\\ &-\sum_{j=0}^\infty \left(-\frac{c_1t^{\alpha_2-\alpha_1}}{c_2}\right)^{j+1} E^{j+1}_{\alpha_2,(\alpha_2-\alpha_1)(j+1)+1}\left(-\frac{ut^{\alpha_2}}{c_2}\right)\ .\end{aligned} \tag{2.15}$$

Example 2.10. Inverse tempered α-stable subordinator. Let $D(t)$ be a tempered α-stable subordinator with Lévy exponent $\psi_D(k) = (k+\vartheta)^\alpha - \vartheta^\alpha$ with $\vartheta > 0$ and $\alpha \in (0,1)$. Then, we have

$$\eta(u,t) = 1 - u\int_0^t \exp(-\vartheta\tau)\tau^{\alpha-1}E^1_{\alpha,\alpha}((\vartheta^\alpha - u)\tau^\alpha)d\tau\ . \tag{2.16}$$

Example 2.11. Inverse Poisson subordinator. Let $D(t)$ be a Poisson subordinator with Lévy exponent $\psi_D(k) = \lambda[1-\exp(-k\Lambda)]$ with $\lambda > 0$ and $\Lambda > 0$. Then, we have

$$\eta(u,t) = \left(1+\frac{u}{\lambda}\right)^{-\left(\lfloor \frac{t}{\Lambda} \rfloor + 1\right)} . \tag{2.17}$$

3. Option Pricing With Fourier Transform Method

To price the options, we need to change the measure from the physical measure P to the risk neutral measure Q. For simplicity, we assume that the stochastic process for the asset price $S(t)$ continues to be an exponential of a subdiffusive Lévy process under the measure Q, that is

$$S(t) = S(0)\exp[rt + \xi(t) + Y(t)] , \tag{3.1}$$

where r is the risk-free interest rate and $Y(t)$ is the subdiffusive Lévy process defined in (2.2). The deterministic function $\xi(t)$ is selected so that the discounted asset price is a nonnegative martingale under the measure Q. We can calculate $\xi(t)$ from the following lemma.

Lemma 3.1. *The function $\xi(t)$ in (3.1) is given by*

$$\xi(t) = -\log[\eta(\psi_L(-1),t)] , \tag{3.2}$$

where $\eta(\cdot,t)$ and $\psi_L(\cdot)$ can be found in (2.12) and (2.4), respectively.

Proof. For the subdiffusive Lévy model to be well defined, $S(t)$ must satisfy the martingale condition:

$$\mathrm{E^Q[S(t)] = \exp(rt)S(0)} .$$

Using the law of iterated expectations and (3.1), we have

$$\mathrm{E^Q[S(t)] = S(0)E^Q[\exp(rt+\xi(t)+Y(t))] = S(0)\exp[rt+\xi(t)]E^Q\{E^Q[\exp(L(T(t)))|T(t)]\}}$$
$$= S(0)\exp[rt+\xi(t)]\mathrm{E^Q[\exp(-\psi_L(-1)T(t))] = S(0)\exp[rt+\xi(t)]\eta(\psi_L(-1),t)} .$$

Therefore, $\xi(t)$ must satisfy (3.2). □

Denote by $\Phi_{\log(S(t))}(u)$ the characteristic function of the log asset price. We can calculate $\Phi_{\log(S(t))}(u)$ from the following lemma.

Lemma 3.2. *For the subdiffusive Lévy process defined in (3.1) and (2.2), the characteristic function $\Phi_{\log(S(t))}(u)$ is given by*

$$\Phi_{\log(S(t))}(u) = \mathrm{E^Q[exp(iu\log(S(t)))] = exp[iu(\log(S(0)) + rt + \xi(t))]}\eta(\psi_{\mathrm{L}}(-\mathrm{iu}),\mathrm{t})\ , \tag{3.3}$$

where $\eta(\cdot,t)$ and $\psi_L(\cdot)$ can be found in (2.12) and (2.4), respectively.

Proof. Using the law of iterated expectations and (3.1), we have

$$\begin{aligned}
&\Phi_{\log(S(t))}(u) = \mathrm{E^Q[exp(iu\log(S(t)))]} \\
&= \exp[iu(\log(S(0)) + rt + \xi(t))]\mathrm{E^Q\{E^Q[exp(iuL(T(t)))|T(t)]\}} \\
&= \exp[iu(\log(S(0)) + rt + \xi(t))]\mathrm{E^Q[exp(-\psi_L(-iu)T(t))]} \\
&= \exp[iu(\log(S(0)) + rt + \xi(t))]\eta(\psi_L(-iu),t)\ .
\end{aligned}$$

□

Let $C(S(0),\tau,K,\beta)$ and $P(S(0),\tau,K,\beta)$ represent the prices of a power call option and a power put option with strike K and maturity τ with payoff functions $(S^\beta(\tau) - K^\beta)^+$ and $(K^\beta - S^\beta(\tau))^+$ for $\beta > 0$, respectively. Note that the power options reduce to vanilla options when $\beta = 1$. The power option prices can be computed from the following proposition.

Proposition 3.3. *For the subdiffusive Lévy model specified in (3.1) and (2.2), we can obtain the power option prices as follows.*

(1) The power call option price $C(S(0),\tau,K,\beta)$ is given by

$$C(S(0),\tau,K,\beta) = \exp(-r\tau)\left[\Phi_{\log(S(\tau))}(-i\beta)P_1 - K^\beta P_2\right]\ , \tag{3.4}$$

where

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathbb{R}\mathrm{e}\left[\frac{\exp(-iu\log(K))f_j(u,\tau)}{iu}\right]du\ ,$$

for $j = 1,2$ and $\Phi_{\log(S(\tau))}(\cdot)$ can be obtained using Lemma 3.2. Furthermore,

$$f_1(u,\tau) = \frac{\Phi_{\log(S(\tau))}(u - i\beta)}{\Phi_{\log(S(\tau))}(-i\beta)}\ ,$$

and

$$f_2(u,\tau) = \Phi_{\log(S(\tau))}(u)\ .$$

(2) The power put option price $P(S(0),\tau,K,\beta)$ is given by

$$P(S(0),\tau,K,\beta) = C(S(0),\tau,K,\beta) - \exp(-r\tau)\left[\Phi_{\log(S(\tau))}(-i\beta) - K^{\beta}\right]\ . \tag{3.5}$$

Proof. The power call option price can be calculated from

$$\begin{aligned} C(S(0),\tau,K,\beta) &= \exp(-r\tau)\mathrm{E}^{\mathrm{Q}}\left[\left(\mathrm{S}^{\beta}(\tau) - \mathrm{K}^{\beta}\right)^{+}\right] \\ &= \exp(-r\tau)\mathrm{E}^{\mathrm{Q}}\left[\mathrm{S}^{\beta}(\tau)1_{\{\mathrm{S}(\tau)>\mathrm{K}\}} - \mathrm{K}^{\beta}1_{\{\mathrm{S}(\tau)>\mathrm{K}\}}\right]\ . \end{aligned}$$

We follow [6] by introducing power numeraire $N^{\beta}(t,\tau)$ as the value at time t of $S^{\beta}(\tau)$ at τ. We also define $\tilde{Q}$ as the martingale measure associated with taking $N^{\beta}(t,\tau)$ as numeraire. Therefore, the Radon-Nikodym derivative of the measure change is given by

$$\frac{d\tilde{Q}}{dQ} = \frac{S^{\beta}(\tau)}{\mathrm{E}^{\mathrm{Q}}[\mathrm{S}^{\beta}(\tau)|\mathscr{F}_{\mathrm{t}}]}$$

on $\mathscr{F}_t$. Then, using change of measures and Lemma 3.2, we have

$$\begin{aligned} C(S(0),\tau,K,\beta) &= \exp(-r\tau)\mathrm{E}^{\mathrm{Q}}\left[\mathrm{S}^{\beta}(\tau)1_{\{\mathrm{S}(\tau)>\mathrm{K}\}} - \mathrm{K}^{\beta}1_{\{\mathrm{S}(\tau)>\mathrm{K}\}}\right] \\ &= \exp(-r\tau)\left\{\mathrm{E}^{\mathrm{Q}}[\mathrm{S}^{\beta}(\tau)]\mathrm{E}^{\tilde{\mathrm{Q}}}(1_{\{\mathrm{S}(\tau)>\mathrm{K}\}}) - \mathrm{K}^{\beta}\mathrm{E}^{\mathrm{Q}}[1_{\{\mathrm{S}(\tau)>\mathrm{K}\}}]\right\} \\ &= \exp(-r\tau)\left[\Phi_{\log(S(\tau))}(-i\beta)P_1 - K^{\beta}P_2\right]\ , \end{aligned}$$

with two probabilities P_1 and P_2, which can be calculated from the corresponding characteristic functions f_1 and f_2.

$$\begin{aligned} f_1(u,\tau) &= \mathrm{E}^{\tilde{\mathrm{Q}}}[\exp(\mathrm{iu}\log(\mathrm{S}(\tau)))] \\ &= \frac{1}{\mathrm{E}^{\mathrm{Q}}[\mathrm{S}^{\beta}(\tau)]}\mathrm{E}^{\mathrm{Q}}[\exp(\mathrm{iu}\log(\mathrm{S}(\tau)))\mathrm{S}^{\beta}(\tau)] \\ &= \frac{\Phi_{\log(S(\tau))}(u-i\beta)}{\Phi_{\log(S(\tau))}(-i\beta)}\ , \end{aligned}$$

and

$$f_2(u,\tau) = \mathrm{E}^{\mathrm{Q}}[\exp(\mathrm{iu}\log(\mathrm{S}(\tau)))]$$
$$= \Phi_{\log(S(\tau))}(u) \ .$$

By inverting two characteristic functions f_1 and f_2, we can obtain two probabilities P_1 and P_2 through

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathbb{R}\mathrm{e}\left[\frac{\exp(-iu\log(K))f_j(u,\tau)}{iu}\right] du \ ,$$

for $j = 1, 2$.

The power put option price can be easily obtained through put-call parity for power options.

□

When there is no time change, i.e. $T(t) = t$, the subdiffusive Lévy model becomes Lévy model. If in addition, the underlying Lévy process $L(t)$ is a Brownian motion, we obtain the Black-Scholes model. The power call option price under the Black-Scholes model can be calculated using the following result (see e.g. [13]).

Proposition 3.4. *Assume the process for the asset price under the risk neutral measure Q is*

$$S(t) = S(0)\exp\left[\left(r - \frac{\sigma^2}{2}\right)t + \sigma B(t)\right] \ ,$$

where $B(t)$ is a Brownian motion. The power call option price is given by

$$C(S(0),\tau,K,\beta) = S^\beta(0)\exp\left[(\beta-1)\left(r + \frac{\beta\sigma^2}{2}\right)\tau\right]N(d_1) - K^\beta\exp(-r\tau)N(d_2) \ ,$$

where $N(\cdot)$ is the CDF of a standard normal distribution. Furthermore,

$$d_1 = \frac{\log\frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2 + \beta\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \beta\sigma\sqrt{\tau} \ .$$

4. A Numerical Study

In this section, we numerically study four different subdiffusive models:

- A subdiffusive Lévy model where $L(t)$ is Gaussian and $T(t)$ is an inverse α-stable subordinator;

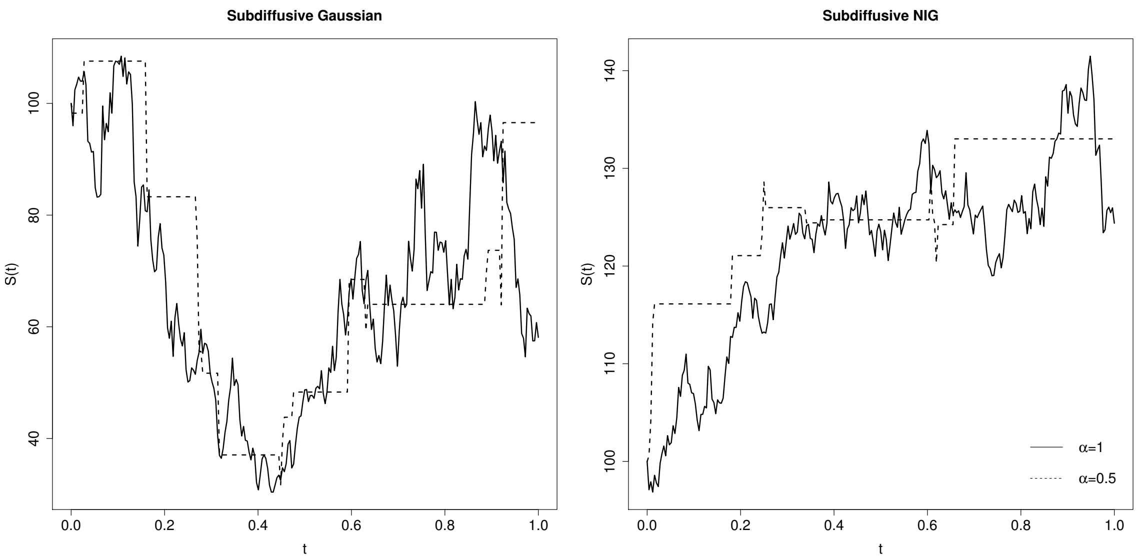

FIGURE 1. Simulated $S(t)$ for subdiffusive Lévy processes with inverse α-stable subordinator. For Gaussian process: $\mu = 0.05$, $\sigma = 0.3$. For NIG process: $\mu = 0.05$, $\sigma = 0.2$, $a = 6$, $b = -4$, $\delta = 0.1$.

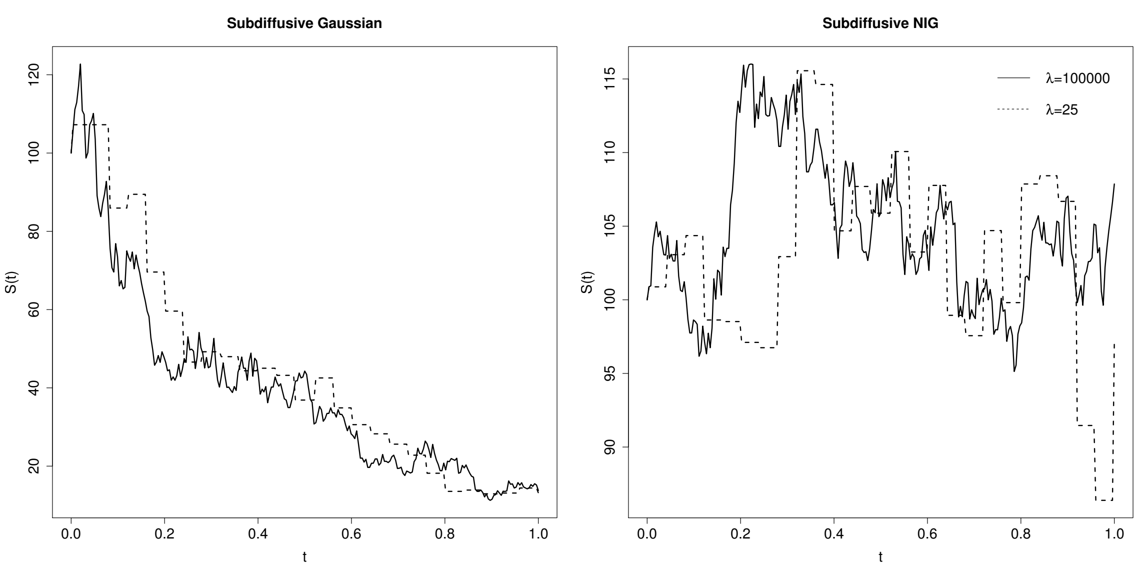

FIGURE 2. Simulated $S(t)$ for subdiffusive Lévy processes with inverse Poisson subordinator. For Gaussian process: $\mu = 0.05$, $\sigma = 0.3$. For NIG process: $\mu = 0.05$, $\sigma = 0.2$, $a = 6$, $b = -4$, $\delta = 0.1$. For inverse Poisson subordinator: $\Lambda = 1/\lambda$.

- A subdiffusive Lévy model where $L(t)$ is NIG and $T(t)$ is an inverse α-stable subordinator;
- A subdiffusive Lévy model where $L(t)$ is Gaussian and $T(t)$ is an inverse Poisson subordinator;
- A subdiffusive Lévy model where $L(t)$ is NIG and $T(t)$ is an inverse Poisson subordinator.

The Laplace transforms for inverse α-stable subordinator and inverse Poisson subordinator are given in (2.14) and (2.17), respectively. We note that for the subdiffusive Lévy model with inverse α-stable subordinator, the model converges to the Lévy model without subdiffusion when $\alpha \to 1$. Similarly, for the subdiffusive Lévy model with inverse Poisson subordinator, the model converges to the Lévy model when $\lambda \to \infty$.

In Figure 1, we simulate a path of subdiffusive Lévy processes with inverse α-stable subordinator. We plot the simulated path for $\alpha = 0.5$ and $\alpha = 1$, where the latter represents the corresponding Lévy processes without subdiffusion. In Figure 2, we plot the simulated path for subdiffusive Lévy processes with inverse Poisson subordinator. For comparison purpose, we plot the path for both $\lambda = 25$ and $\lambda = 100000$, where the latter converges to the corresponding Lévy processes without subdiffusion.

It is clear for all the subdiffusive Lévy processes considered, there are flat periods of trajectories that correspond to the periods when the inverse subordinator $T(t)$ is constant. We also note that inverse α-stable subordinator and inverse Poisson subordinator describe two different kinds of illiquidity. The inverse α-stable subordinator can replicate prolonged periods of inactivity whereas the inverse Poisson subordinator tends to produce the recurring illiquidity at high or low frequency. The difference in the behavior of the two inverse subordinators is due to the difference in the Lévy subordinators that governs the probability law for the length of constant periods.

In Figures 3-6, we display the implied volatility smile patterns with respect to the strike price K for power call options with $\beta = 2$ under subdiffusive Lévy models. We also compare the implied volatilities obtained from the subdiffusive Lévy models with the corresponding Lévy models. Although for all the models considered, the conditional expectation of asset prices stays

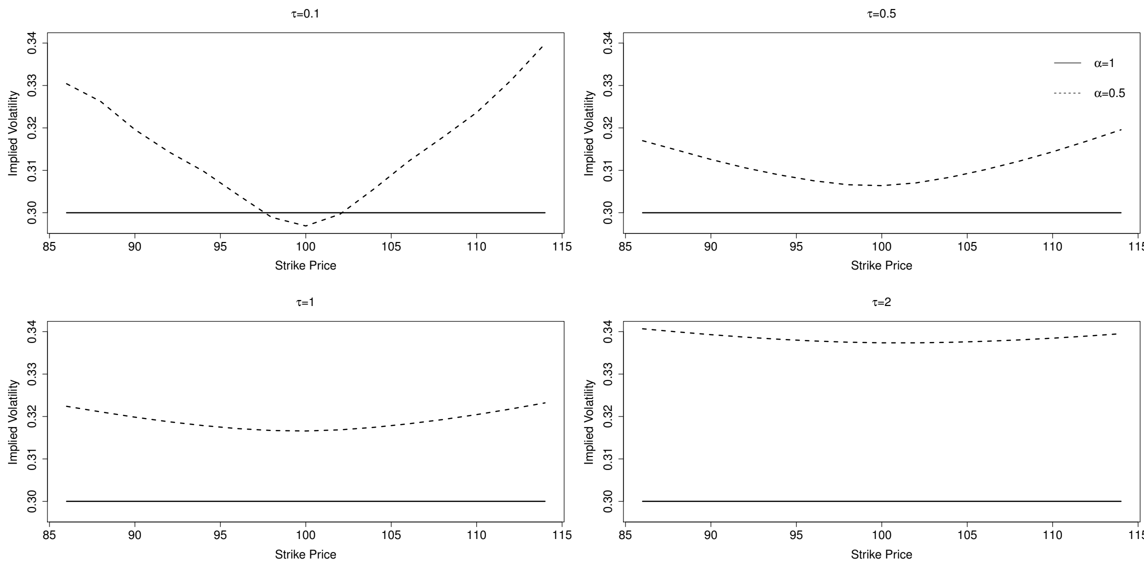

FIGURE 3. Implied volatilities for subdiffusive Gaussian processes with inverse α-stable subordinator for different strike prices K. $\mu = 0$, $\sigma = 0.3$, $r = 0.05$, $S(0) = 100$ and $\beta = 2$.

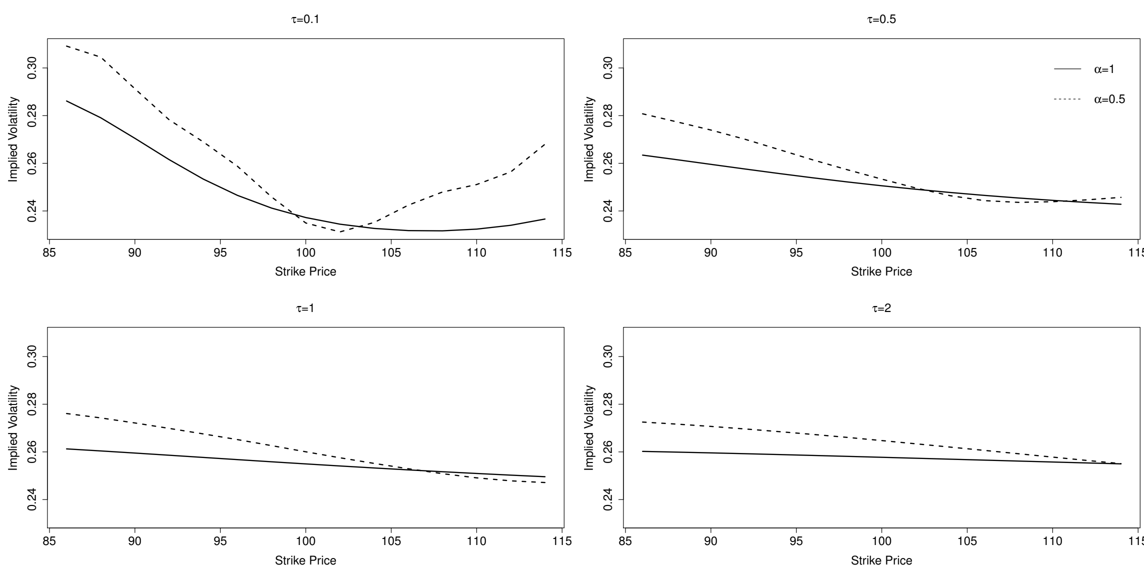

FIGURE 4. Implied volatilities for subdiffusive NIG processes with inverse α-stable subordinator for different strike prices K. $\mu = 0$, $\sigma = 0.2$, $a = 6$, $b = -4$, $\delta = 0.1$, $r = 0.05$, $S(0) = 100$ and $\beta = 2$.

the same, the conditional volatility varies with α for inverse α-stable subordinator and λ for inverse Poisson subordinator. When the underlying process is Gaussian without subdiffusion, the implied volatility always is a constant. However, once the Gaussian process is time changed

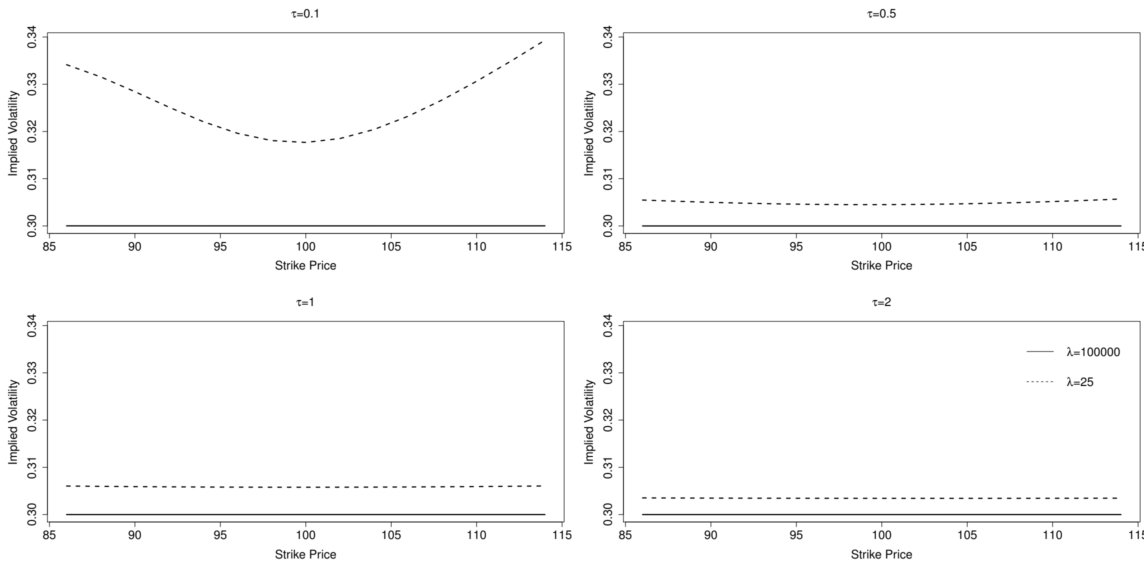

FIGURE 5. Implied volatilities for subdiffusive Gaussian processes with inverse Poisson subordinator for different strike prices K. $\mu = 0$, $\sigma = 0.3$, $\Lambda = 1/\lambda$, $r = 0.05$, $S(0) = 100$ and $\beta = 2$.

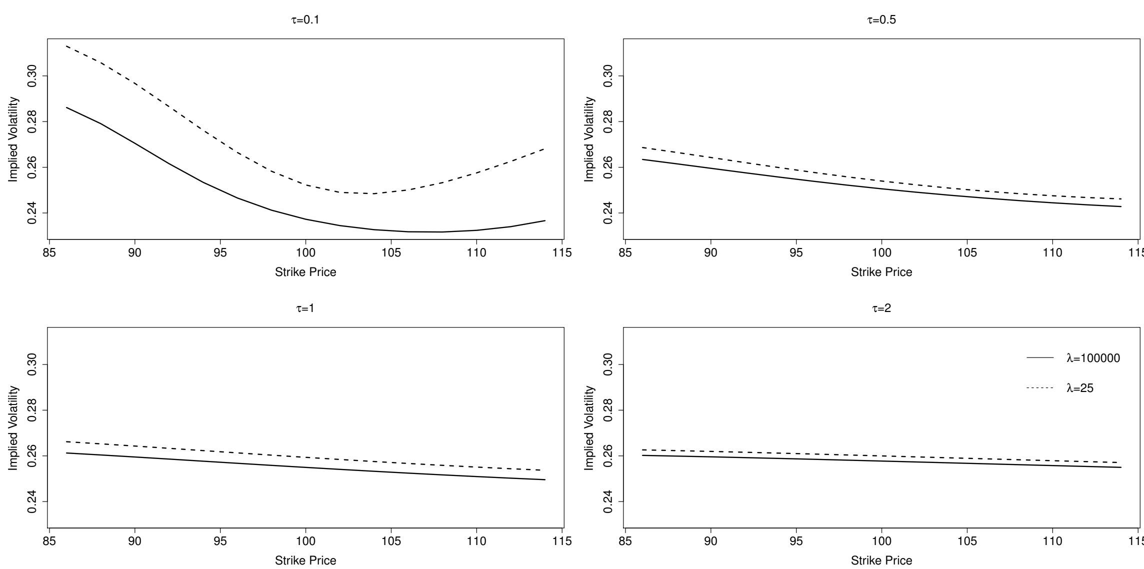

FIGURE 6. Implied volatilities for subdiffusive NIG processes with inverse Poisson subordinator for different strike prices K. $\mu = 0$, $\sigma = 0.2$, $a = 6$, $b = -4$, $\delta = 0.1$, $\Lambda = 1/\lambda$, $r = 0.05$, $S(0) = 100$ and $\beta = 2$.

by either inverse α-stable subordinator or inverse Poisson subordinator, the volatility smile can be produced. When the underlying Lévy process is an NIG process, more complex smile patterns can be generated through subdiffusion. We also observe when the maturity increases, the term structure of implied volatilities flattens out for all the models. However, it seems that the speed of flattening out is slower for subdiffusive models, which indicates the subdiffusive models have the potential to replicate the term structure of implied volatility better than the Lévy models.

5. CONCLUSION

The Lévy model generalizes the famous Black-Scholes model and is able to capture the volatility skew often observed in equity option prices. For illiquid markets where the number of participants and thus the number of transactions is low, the asset prices can exhibit constant time periods. To capture this subdiffusion behaviour, we extend the Lévy model to a subdiffusive Lévy model, where the underlying Lévy process is time changed by a general inverse Lévy subordinator. We are able to derive the formula for the characteristic function of the log asset price by realizing the fact that the Laplace transform for commonly encountered inverse subordinators is known explicitly. Without numerical methods such as Monte Carlo or PDE, we obtain the analytical solution to the power option prices through Fourier transform. We also numerically study the behaviour of the newly proposed models by comparing them with the existing Lévy models.

In this paper, for illustration purposes, the option prices are computed through inverting the characteristic functions numerically. The more efficient methods such as Fast Fourier transform of [1] or Fourier cosine method of [2] can also be employed. Furthermore, it will be an interesting topic to study the pricing problems for other exotic options or American options under subdiffusive Lévy model.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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