Available online at http://scik.org
Math. Finance Lett. 2021, 2021:3
https://doi.org/10.28919/mfl/5989
ISSN: 2051-2929

PRECISE CONTINUOUS TIME EXPECTED DEFLATOR FOR THE G2++ MODEL

FRANCESCO STRATI[1,*], LUCA G. TRUSSONI[2]

[1]KPMG Advisory SpA, Actuarial Services, Milan, Italy

[2]LTLogics, Milan, Italy

Abstract. We propose a continuous time algorithm for the estimation of the cash account stochastic discount factor computed by Monte-Carlo simulations of a G2++ model. By taking into account previous results in a discrete time framework, we define a new algorithm that is *i*) easier to implement without loss of precision and *ii*) can be used in continuous time models without incurring in possible discretization errors. The simulated stochastic discount factor will be precise in terms of the risk-free measure adopted even when the number of simulated paths is small.

Keywords: statistical simulation methods; bond interest rates; actuarial studies.

2010 AMS Subject Classification: 91G60, 62P05.

1. Introduction

In Strati and Trussoni [3] it has been defined an algorithm by which it is possible to simulate precise expected stochastic discount factors, even with a small number of simulations, by studying the transition law of the discrete random variable

$$\int_{[t,t+\Delta t]} x(\tau)+y(\tau)d\tau$$

in the time span $[t,t+\Delta t]$ conditional on the observation at time t. The two stochastic factors $x(\cdot)$ and $y(\cdot)$ are modeled in a Hull-White two factor interest rate model, as proposed by Brigo

*Corresponding author

E-mail address: fstrati@kpmg.it

Received May 7, 2021

and Mercurio [1] (G2++ model). In this paper we shall define an algorithm that can simulate precise expected stochastic discount factors by exploiting the law of the continuous random variable

$$\int_t^T [x(s)+y(s)]ds,$$

conditional to the σ-field $\mathscr{F}_t$ (see Brigo and Mercurio [1]).

The G2++ model is a short-rate model in which the instantaneous interest rate process is defined by the sum of two correlated Ornstein-Uhlenbeck processes and a deterministic function of time calibrated in order to exactly fit the current term structure of discount factors (Brigo and Mercurio [1]). Our purpose is to exploit the joined transition law of the two O-U processes so as to obtain a precise expected value of the deflator even when the number of Monte-Carlo simulations is small. It will be of utmost importance for models that cannot allow for a huge number of simulations n, since the precision of Monte Carlo methods depends on the minimization of the standard error with a convergence rate of $\mathscr{O}(n^{-1/2})$: a small number of paths leads to relevant simulation errors. This approach finds natural applications in the insurance industry since, by effect of Solvency II directive, insurers must use financial stochastic models to cope with policy rates whose interest rate risk is not transferred to clients (for example the valuation of Italian or French participating life insurance policies, embedded with options and financial guarantees). We shall measure the precision of the continuous time algorithm presented here by means of methods commonly used among practitioners in life insurance stochastic modeling to evaluate liabilities linked to segregated funds in presence of minimum interest rate guarantee; these models are known as ”Economic Scenario Generators”. Another reason to consider application in insurance industry is that the official deterministic risk-free interest rates term structure is monthly provided by the European Insurance and Occupational Pensions Authority (EIOPA), and it is possible to compare to it the expected deflator calculated from simulations.

The algorithm shown here is easier to implement with respect to the discrete case (see Strati and Trussoni [3]) without any significant observed loss of precision, is free from discretization errors, and allows to keep low the number of simulations.

In order to see some issues on the nowadays used insurance models see Vedani et al. [4].

2. G2++ SIMULATED DEFLATOR

The two factors G2++ model of Brigo and Mercurio [1], is defined by a sum of two correlated Ornstein-Uhlenbeck processes ($x(t)$ and $y(t)$, called stochastic factors) and a deterministic shift ($\varphi(t)$) added so as to fit exactly the initial zero-coupon curve set up by considering the most liquid market data (see Vedani et al. [4] and Moreni and Pallavicini [2] on this topic). The instantaneous short-rate process under the risk-neutral measure $\mathbb{Q}$ is

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0, \quad x(0) = y(0) = 0$$

in which the two Ornstein-Uhlenbeck processes follow two stochastic differential equations

$$dx(t) = -ax(t)dt + \sigma W_1(t)$$

$$dy(t) = -by(t)dt + \eta W_2(t)$$

where (W_1, W_2) are bi-dimensional Brownian motions with correlation ρ

$$dW_1(t)dW_2(t) = \rho dt$$

with $[a,\, b,\, \sigma,\, \eta] > 0$ and $\rho \in [-1, 1]$. The initial value of the deterministic function φ is the current short rate $\varphi(0) = r_0$. The short rate is the instantaneous spot rate, which represents the initial point of the yield curve at each time t. The short rate is moved by the two correlated brownian motions in order to model interest rate curve movement better than a single factor model could do (leading to more accurate pricing of assets whose price is influenced by correlated movement of interest rates at different maturities). The short rate is given by

$$r(t) = x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \\ + \sigma \int_s^t e^{-a(t-u)} dW_1(u) + \eta \int_s^t e^{-b(t-u)} dW_2(u) + \varphi(t).$$

By considering the probability measure $\mathbb{Q}$ that is conditional on the filtration $\mathscr{Q}_s$, it is possible to define the expected value and the variance of the interest rate:

$$\mathbb{E}^{Q}[r(t)|\mathscr{Q}_s] = x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \varphi(t);$$

$$\mathrm{Var}^{Q}\{r(t)|\mathscr{Q}_s\} = \frac{\sigma^2}{2a}[1 - e^{-2a(t-s)}] + \frac{\eta^2}{2b}[1 - e^{-2b(t-s)}] + 2\rho\frac{\sigma\eta}{a+b}[1 - e^{-(a+b)(t-s)}].$$

The dynamics of $r(t)$ can be defined in terms of independent Brownian motions $(\tilde{W}_1, \tilde{W}_2)$ by using the Cholesky decomposition

$$dx(t) = -ax(t)dt + \sigma d\tilde{W}_1(t), \;\; x(0) = 0,$$
$$dy(t) = -by(t)dt + \eta\rho d\tilde{W}_1(t) + \eta\sqrt{1-\rho^2}d\tilde{W}_2(u), \;\; y(0) = 0.$$

for which

$$dW_1(t) = d\tilde{W}_1(t),$$
$$dW_2(t) = \rho d\tilde{W}_1(t) + \sqrt{1-\rho^2}d\tilde{W}_2(t),$$

by integrating over $dx(t)$ and $dy(t)$, as in the dependent case, it is obtained

$$r(t) = x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \sigma \int_s^t e^{-a(t-u)}d\tilde{W}_1(u) + \\ + \eta\rho \int_s^t e^{-b(t-u)}d\tilde{W}_1(u) + \eta\sqrt{1-\rho^2}\int_s^t e^{-b(t-u)}d\tilde{W}_2(u) + \varphi(t).$$

Having defined $r(t)$ (dependent on $x(s)$ and $y(s)$), denote by $P(t,T)$ the price at time t of a zero-coupon bond maturing at T with unit face value, that is

$$P(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r(s)ds\right)\right]. \tag{1}$$

It is fundamental to compute the expectation under the risk-neutral measure of Eq. (1) by means of the normally distributed random variable

$$I(t,T) = \int_t^T [x(s)+y(s)]ds,$$

conditional to the σ-field $\mathscr{F}_t$. Define now the two integrals following the random variable: J_x as for the stochastic factor $x(\cdot)$ and J_y as for the stochastic factor $y(\cdot)$:

$$J_x = \int_t^T x(s)ds; \quad J_y = \int_t^T y(s)ds$$

After some manipulations and integrating by parts (see Brigo and Mercurio [1], Appendix A: Proof of Lemma 4.2.1), it will be obtained

$$\int_t^T x(s)ds = \frac{1-e^{-a(T-t)}}{a}x(t) + \frac{\sigma^2}{a^2}\int_t^T (1-e^{-a(T-s)})^2, \tag{2}$$

for which $J_x = \int_t^T x(s)ds$. The same should be followed when computing J_y by using b, η and W_2. In insurance industry such integral is needed at yearly steps: on the contrary of the discrete algorithm proposed in Strati and Trussoni [3], that relied on short span integrations steps, we propose to use the above formulae (Eq. (2)) applied to long steps (typically, one year long). The algorithm we propose to calculate numerically the integral is detailed in the following pseudo-code:

input: $J_x(0) = 0$, $J_y(0) = 0$, $W_1 \wedge W_2 =$ Latin Hypercube Sampling (LHS)

for $t = 1 \dots n$ **do**

$$J_x(t) = J_x(t-1) + x(t-1) * (1 - \exp(-a))/a + W_1(t-1) * \cdots$$
$$\cdots * \sigma/a * \sqrt{(}1 - 2 * (1 - \exp(-a))/a + (1 - \exp(-2*a))/(2*a))$$

$$J_y(t) = J_y(t-1) + y(t-1) * (1 - \exp(-b))/b + W_2(t-1) * \cdots$$
$$\cdots * \sigma/b * \sqrt{(}1 - 2 * (1 - \exp(-b))/b + (1 - \exp(-2*b))/(2*b))$$

end

By this algorithm, the deflator will exploit the transition law of the random variable $I(t,T)$

$$I(t,T)_{\text{Law}} = e^{-J_x - J_y},$$

by which it is possible to define the continuous time expected *adjusted* deflator as

$$\mathbb{E}[D(t,T)] = \varphi(t-1)\frac{I(t,T)_{\text{Law}}}{\bar{I}_{\text{Law}}}, \tag{3}$$

in which $\bar{I}_{\text{Law}}$ is the average value of the realization of $I(t,T)_{Law}$. The adjusted deflator obtained through (3) is the tool we need to simulate interest rate with a small number of Monte-Carlo simulations and long steps, keeping a good fit to the observed spot rate curve.

3. Testing the Algorithm

In the insurance industry, when Economic Scenario Generators (ESG) are implemented to take into account the interest rate risk embedded with participating tariffs with minimum guarantees and optionalities, the average risk-free rate by which the expected stochastic deflator is defined has to be close to the zero-coupon curve provided by EIOPA. In order to be consistent with the results of Strati and Trussoni [3], it has been implemented a G2++ model as defined in Section 2 with 1000 random trajectories of the zero-coupon curve on 30 maturities over 40 years. The relevant zero-coupon curve considered here is provided by EIOPA, it is with volatility adjustment as of December 31st, 2018. Moreover, as for the calibration of the five G2++ parameters, we have used at-the-money swaption normal implied volatilities as of December 31st, 2018 with tenors 1, 2, 3, 4, 7, 10, 15, 20, 25. The parameters obtained from the optimization algorithm are: $a = 0.5675$, $b = 0.0291$, $\eta = 0.0086$, $\sigma = 0.026$ and ρ maintained at -0.75. Denote by EIOPA_r the deterministic risk-free rate term structure. A very simple, but strong test for the precision of the expected deflator is the ratio test (RT):

$$\text{RT} = \frac{\mathbb{E}[D(t,T)]}{\text{EIOPA}_r}.$$

by which RT ≈ 1 for each observed time. The more RT departs from 1, the more the drift of the stochastic model will be distorted. The Ratio Test for each year is plotted in Figure 1. In

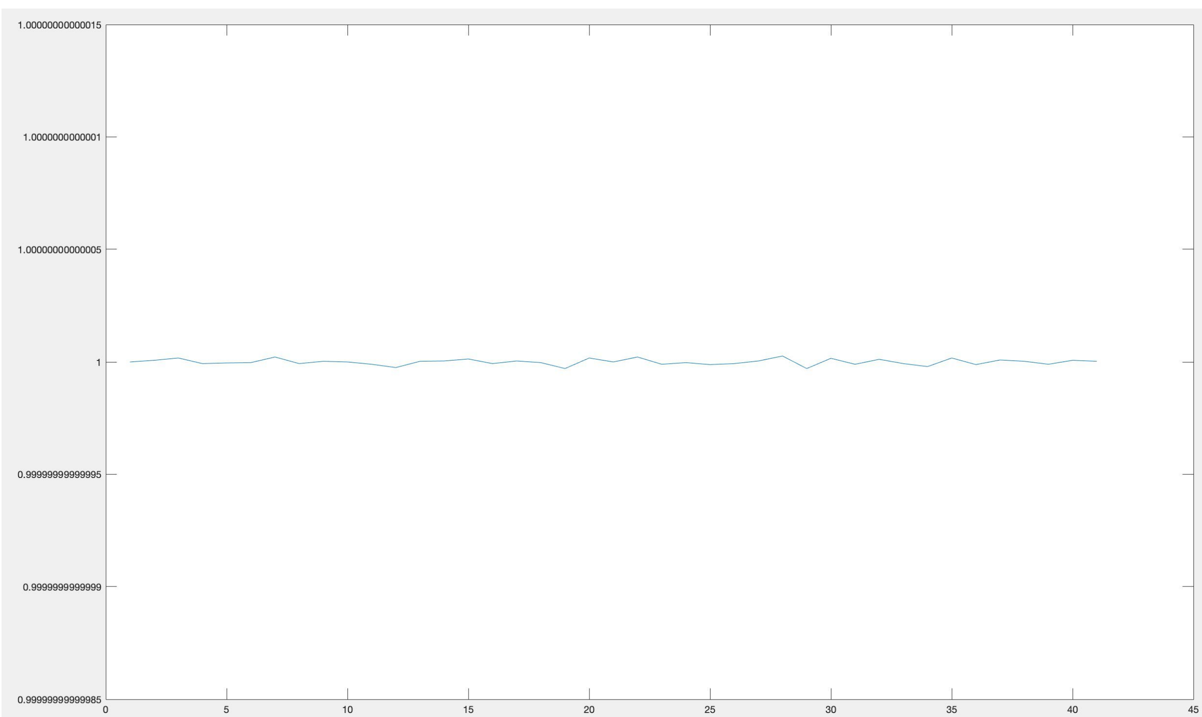

FIGURE 1. Ratio Test Expected Stochastic Discount Factor with continuous time adjustment

general, the expected cash account deflator is defined as

$$\mathbb{E}[P(t,T)_1] = \int_t^T \exp(-r_{1,s}ds)$$

in which $r_{1,s}$ is the stochastic simulation over a time horizon $t \leq s \leq T$ of the first year interest rate in the deterministic term structure. It is what we consider as *not adjusted* stochastic deflator, that is the deflator of the model without the drift-adjustment needed for considering the Monte Carlo approximation error caused by the small number of simulations. The ratio test for the not-adjusted deflator is plotted in Figure 2. As it can be observed, the continuous time algorithm produces a precise expected deflator since RT ≈ 1, while the not-adjusted expected deflator is not well fitted since it decreases RT < 1 after the 25th year of projection. In order

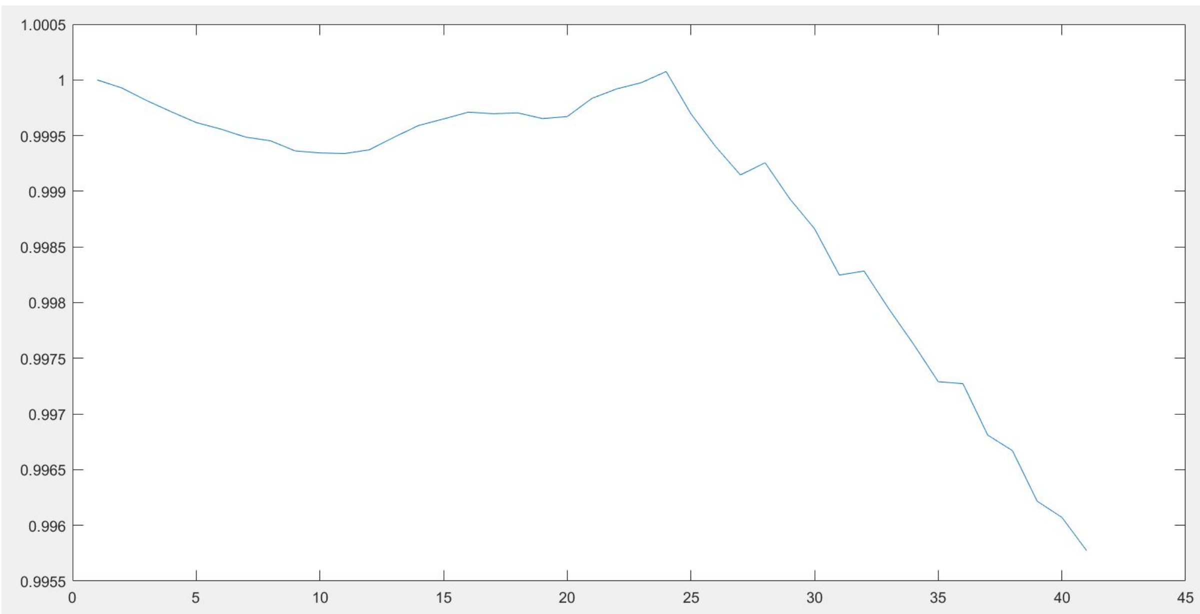

FIGURE 2. Ratio Test Expected Stochastic Discount Factor without adjustment

to see the effects of this error, we shall compare the results of the martingale test of the not-adjusted expected deflator with the continuous time algorithm together with the discrete time algorithm computed in Strati and Trussoni [3], so as to assess if the algorithm presented here is efficient as that defined in discrete form. In particular, insurance practitioners take into account analytical methods for computing the martingale tests. It is possible to check if, considering all the simulations, the evolution of the expected discounted price at time t whose maturity is $K > t$ is close to the evolution of the deterministic risk-free interest rate term structure provided by EIOPA, that is

$$M(t) = \frac{\mathbb{E}[P(t,K) \cdot D(t,K)]}{\text{EIOPA}_r(K+t-1)}$$

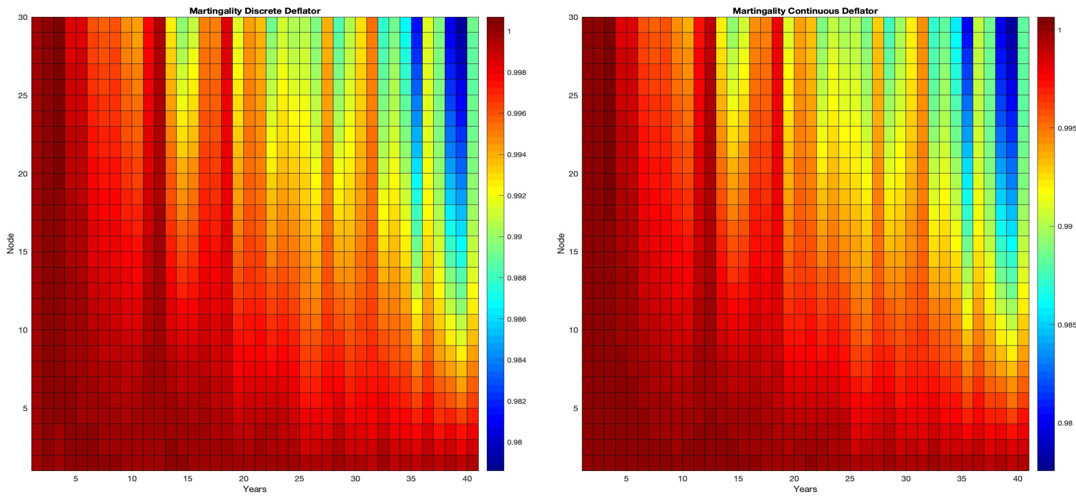

FIGURE 3. Martingale test (Discounted by the discrete (on the left) and continuous time adjusted expected deflator (on the right).

for the adjusted continuous (and discrete) time stochastic mean deflator $\mathbb{E}[D(t,T)]$ as defined in Eq. (3), and

$$G(t) = \frac{\mathbb{E}[P(t,K) \cdot P(t,K)_1]}{\text{EIOPA}_r(K+t-1)}$$

for the not adjusted expected deflator $\mathbb{E}[P(t,T)_1]$. The aim is to obtain $M(t) \approx 1$ and $G(t) \approx 1$. The martingale test for the for continuous and discrete time are plotted in Figure 3. For one thousand simulations, the highest absolute difference $|1-M(t)|$ among all the maturities K and times t, is of $0,023073$ in the continuous case and of $0,022076$ in the discrete case. Moreover, for five hundreds thousand simulations, the continuous time algorithm accounts for the highest error of $0,0019964$, while for the discrete case is of $0,0025601$. The inverted ranking of the highest errors in case of high number of simulations is due to the discretization error that is overcame by the continuous time algorithm that is more precise than the discrete one in case of a high number of simulations.

As for the martingale test with not adjusted stochastic deflators, the highest absolute difference $|1-G(t)|$ is of $1,5014$ and it is a clear evidence of a martingale test failure (see Figure 4).

4. CONCLUSIONS

In this paper we have presented a continuous time algorithm for computing a precise expected deflator even when the number of Monte Carlo simulations is small, by considering a Hull-White two factor interest rate model. The algorithm presented here is easier to implement with

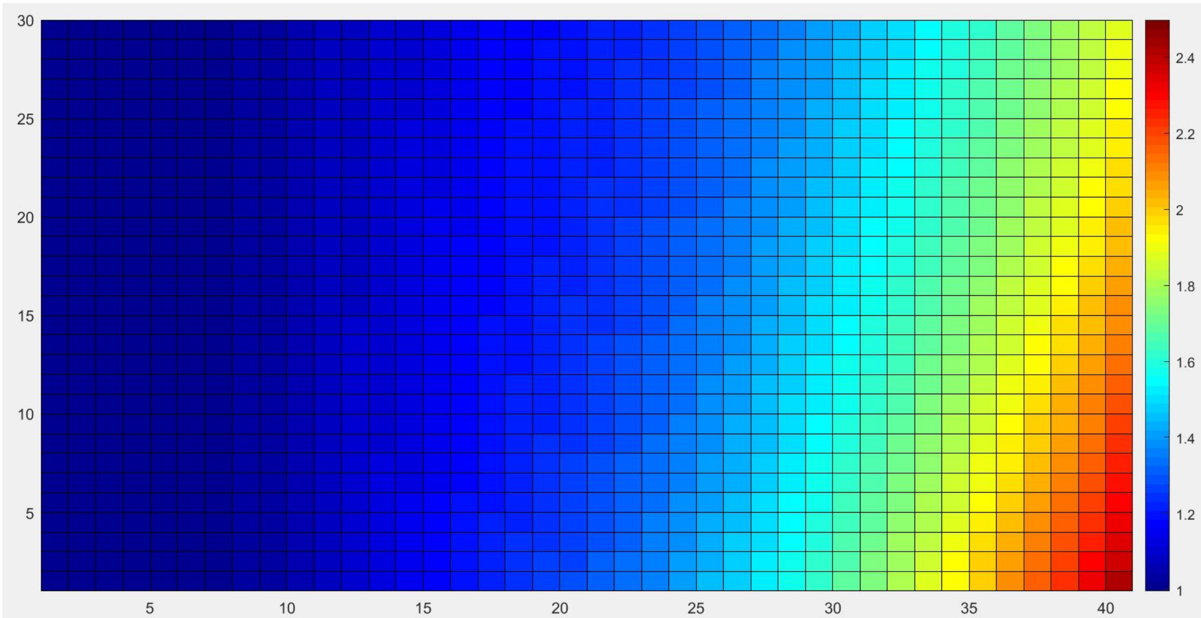

FIGURE 4. Martingale test (Discounted by not adjusted expected deflator).

respect to the discrete time version of Strati and Trussoni [3], and it shows a better precision when the number of simulations increases. The natural application of the algorithm is in the implementation of ESGs used in insurance industry for the evaluation of liability portfolios in which the interest rate risk is not transferred to clients (i.e. in presence of a minimum guaranteed interest rate or other optionalities).

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] B. Damiano, F. Mercurio, Interest Rate Models: Theory and Practice with Smile, Inflation and Credit, Springer, Berlin Heidelberg, Germany, 2004.

[2] N. Moreni, A. Pallavicini, Parsimonious HJM modelling for multiple yield curve dynamics, Quant. Finance 14 (2014), 199-210.

[3] F. Strati, L. G. Trussoni, Exact cash-account deflator for the G2++ Model, Int. J. Financial Eng. 7 (2020), 2050009.

[4] J. Vedani, N. El Karoui, S. Loisel and J. L. Prigent, Market inconsistencies of market-consistent European life insurance economic valuations: pitfalls and practical solutions, Eur. Actuar. J. 7 (2017), 1-28.