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OPTIMAL PORTFOLIO AND WEALTH VALUATION STRATEGIES WITH STOCHASTIC CASH FLOWS DEPENDENT ON THE OPTIMAL WEALTH

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Abstract. This paper examines and study the optimal portfolio and wealth valuation involving stochastic cash inflows and stochastic cash outflows for a certain investor who trades in a complete diffusion models, receives a stochastic cash inflows and pays a stochastic outflows. The dynamics of the wealth process is assumed to involved two risky assets and a cash account. We established the optimal value of wealth and show that the cash inflows and cash outflows depend on the optimal wealth of the investor.

Keywords: optimal portfolio; stochastic; cash inflows; cash outflows; wealth valuation; investor.

2000 AMS Subject Classification: 91B28; 91B30; 91B70; 93E20

1. Introduction

This paper consider the optimal portfolios and wealth valuation of an investor, who receive continuous-time stochastic cash inflows and pays continuously a stochastic cash outflows. The cash inflows are invested into a cash account, an inflation-linked bond and a stock. A compact form of the value of wealth process of the investor was established in this paper. The aim of this paper is to determine the optimal portfolios (in stock and

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inflation-linked bond) and the terminal value of wealth that will accrued to the investor. The cash inflows is found to depends only on the risks associated with the financial market while the cash outflows depend on its own risks and the risk associated with the financial market.

In related literature, Davis [1] examined the rationale, nature and financial consequences of two alternative approaches to portfolio regulations for the long-term institutional investor sectors of life insurance and pension funds. Brawne *et al* [2] developed a model for analyzing the *ex ante* liquidity premium demanded by the holder of an "illiquid annuity". The annuity is an insurance product that is similar to a pension savings account with both an accumulation and "decumulation" phase. They computed the yield needed to compensate for the utility welfare loss, which is induced by the inability to re-balance and maintain an optimal portfolio when holding an annuity. Deelstra *et al* ([3],[4],[5]) considered the optimal design of the minimum guarantee in a defined contribution pension fund scheme. They studied the investment in the financial market by assuring that the pension fund optimizes its retribution which is a part of the surplus, that is the difference between the pension fund value and the guarantee. Zhang *et al* [6] considered the optimal management and inflation protection strategy for defined contribution pension plans using Martingale approach. They derived an analytical expression for the optimal strategy and expresses it in terms of observable market variables. Our aim is to determine the portfolio values and value wealth for the investor.

Nkeki [7], and Nkeki and Nwozo [8] studied the variational form of classical portfolio strategy and expected wealth for a pension plan member. They assumed that the growth rate of salary is a linear function of time and that the cash inflow is stochastic. Nkeki and Nwozo [9], studied the optimal portfolio strategies with stochastic cash flows and expected optimal terminal wealth under inflation protection for a certain Investment Company who trades in a complete diffusion model, receives a stochastic cash inflows and pays a stochastic outflows to its holder. They found that as the market evolve parts of the inflation-linked bond and stock portfolio values should be transferred to cash account. This, to a great extent will protect the IC from catastrophic fall in the stock market. They also found that the portfolio processes involved inter-temporal hedging terms that offset any shock to both the stochastic cash inflows and cash outflows.

The remainder of this paper is organized as follows. In section 2, we present the financial market models. The wealth process of the investor is presented in section 3. Section 4 presents the expected value of discounted cash inflows and cash outflows of the investor. The wealth valuation of the investor is presented in section 5. In section 6, we present the optimal portfolio strategies for the investor. In section 7, we present the value of investor's wealth and cash flows that dependent on optimal wealth of the investor. Finally, section 8 conclude the paper.

2. The Financial Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathbb{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0, T]\}$, where $\mathcal{F}_t = \sigma(S(s), I(s) : s \leq t)$, where $S(t)$ is stock price process at time $s \leq t$, $I(t)$ is the inflation index at time $s \leq t$. The Brownian motions $W(t) = (W^I(t), W^S(t))'$, $0 \leq t \leq T$ is a 2-dimensional process, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}(\mathcal{F}), \mathbb{P})$, where $\mathbb{P}$ is the real world probability measure and σ^S and σ_I are the volatility of stock and volatility of the inflation-linked bond with respect to changes in $W^S(t)$ and $W^I(t)$, respectively. μ is the appreciation rate for stock. Moreover, σ^S and σ_I are the volatilities for the stock and inflation-linked bond respectively, referred to as the coefficients of the market and are progressively measurable with respect to the filtration $\mathbb{F}$.

We assume that the investor faces a market that is characterized by a risk-free asset (cash account) and two risky assets, all of whom are tradeable. In this paper, we allow the stock price to be correlated to inflation. Also, we correlated the cash inflows and outflows to stock market in other to determine the extent to which cash inflows and outflows should be hedged. The dynamics of the underlying assets are given by (1) to (3)

$$dB(t) = rB(t)dt, B(0) = 1; \tag{1}$$

$$dS(t) = \mu S(t)dt + \sigma_1^S S(t)dW^I(t) + \sigma_2^S S(t)dW^S(t), S(0) = s_0 > 0; \tag{2}$$

$$dF(t, I(t)) = (r + \sigma_1\theta^I)F(t, I(t))dt + \sigma_I F(t, I(t))dW(t), F(0) = F_0 > 0; \tag{3}$$

where, r is the nominal interest rate, θ^I is the price of inflation risk, $B(t)$ is the price process of the cash account at time t, $S(t)$ is stock price process at time t, $I(t)$ is the inflation index at time t and has the dynamics:
$dI(t) = E(q)I(t)dt + \sigma_1 I(t)dW^I(t)$,
where $E(q)$ is the expected rate of inflation, which is the difference between nominal interest rate, r and real interest rate R (i.e. $E(q) = r - R$). $F(t, I(t))$ is the inflation-indexed bond price process at time t and $\sigma_I = (\sigma_1, 0)$.
Then, the volatility matrix

$$\Sigma := \begin{pmatrix} \sigma_1 & 0 \\ \sigma_1^S & \sigma_2^S \end{pmatrix} \tag{4}$$

corresponding to the two risky assets and satisfies $det(\sigma) = \sigma_1\sigma_2^S \neq 0$. Therefore, the market is complete and there exists a unique market price θ satisfying

$$\theta := \begin{pmatrix} \theta^I \\ \theta^S \end{pmatrix} = \begin{pmatrix} \theta^I \\ \dfrac{\mu - r - \theta^I\sigma_1^S}{\sigma_2^S} \end{pmatrix} \tag{5}$$

where θ^S is the market price of stock risks and θ^I is the market price of inflation risks (MPIR). We now define the following exponential process which we assume to be Martingale in $\mathbb{P}$:

$$Z(t) = \exp(-\theta' W(t) - \frac{1}{2}\|\theta\|^2 t), \tag{6}$$

where, $W(t) = (W^I(t), W^S(t))'$. We assume in this paper that the cash inflows process $\varphi(t)$ at time t and cash outflows process $\Gamma(t)$ at time t follow the dynamics, respectively presented in (7) and (8).

$$d\varphi(t) = \varphi(t)(\omega dt + \sigma_\varphi dW(t)), \varphi(0) = \varphi_0 > 0 \tag{7}$$

$$d\Gamma(t) = \Gamma(t)(\delta dt + \sigma_\Gamma dW(t)), \Gamma(0) = \Gamma_0 > 0 \tag{8}$$

where, $\sigma_\varphi = (\sigma_1^\varphi, \sigma_2^\varphi)$, $\sigma_\Gamma = (\sigma_1^\Gamma, \sigma_2^\Gamma)$, $\omega > 0$ is the expected growth rate of the cash inflows and σ_1^φ is the volatility caused by the source of inflation, $W^I(t)$ and σ_2^φ is the volatility caused by the source of uncertainty arises from the stock market, $W^S(t)$, and $\delta > 0$ is the expected growth rate of the cash outflows and σ_1^Γ is the volatility caused by the source of inflation, $W^I(t)$ and σ_2^Γ is the volatility caused by the source of uncertainty arises from the stock market, $W^S(t)$.

Remark 1. *The cash inflows can be seen as the contribution of a pension plan member in pension fund and the cash outflows, the minimum benefit that will accrued to pension plan member.*

Applying *Itô* Lemma on (7) and (8), we have the following

$$\varphi(t) = \varphi_0 \exp((\omega - \frac{1}{2}\|\sigma_\varphi\|^2)t + \theta' W(t)), \tag{9}$$

$$\Gamma(t) = \Gamma_0 \exp((\delta - \frac{1}{2}\|\sigma_\Gamma\|^2)t + \theta' W(t)). \tag{10}$$

3. The Wealth Process

Let $X^{\varphi\Gamma}(t)$ be the wealth process at time t, where $\Delta(t) = (\Delta^I(t), \Delta^S(t))$ is the portfolio process at time t and $\Delta^I(t)$ is the proportion of wealth invested in the inflation-linked bond at time t and $\Delta^S(t)$ is the proportion of wealth invested in stock at time t. Then, $\Delta_0(t) = 1 - \Delta^I(t) - \Delta^S(t)$ is the proportion of wealth invested in cash account at time t.

Definition 1. *The portfolio process Δ is said to be self-financing if the corresponding wealth process $X(t)$, $t \in [0, T]$, satisfies*

$$\begin{aligned}
&dX^{\varphi\Gamma}(t) = \Delta^S(t) X^{\varphi\Gamma}(t) \frac{dS(t)}{S(t)} + \Delta^I(t) X^{\varphi\Gamma}(t) \frac{dF(t, I(t))}{F(t, I(t))} + \\
&(1 - \Delta^S(t) - \Delta^I(t)) X^{\varphi\Gamma}(t) \frac{dB(t)}{B(t)} + (\varphi(t) - \Gamma(t))dt, \\
&X^{\varphi\Gamma}(0) = x_0.
\end{aligned} \tag{11}$$

(11) can be re-written in compact form as follows:

$$\begin{aligned}
&dX^{\varphi\Gamma}(t) = X^{\varphi\Gamma}(t)[(r + \Delta(t)A)dt + (\Sigma\Delta'(t))' dW(t)] + (\varphi(t) - \Gamma(t))dt, \\
&X^{\varphi\Gamma}(0) = x_0,
\end{aligned} \tag{12}$$

where, $A = (\sigma_1\theta^I, \mu - r)'$ and $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_1^S & \sigma_2^S \end{pmatrix}$. If the wealth process does not involve cash outflows, then, (12) becomes

$$\begin{aligned} &dX^\varphi(t) = X^\varphi(t)[(r + \Delta(t)A)dt + (\Sigma\Delta'(t))'dW(t)] + \varphi(t)dt, \\ &X^\varphi(0) = x_0. \end{aligned} \tag{13}$$

If the wealth process does not involve cash inflows and cash outflows, then, (12) becomes

$$\begin{aligned} &dX(t) = X(t)[(r + \Delta(t)A)dt + (\Sigma\Delta'(t))'dW(t)], \\ &X(0) = x_0. \end{aligned} \tag{14}$$

4. The Expected Value of Discounted Cash Inflows and Outflows

In this section, we consider the valuation of discounted future cash inflows and cash outflows at time t.

4.1. The Expected Value of Discounted Cash Inflows

Definition 2. *The expected value of discounted future cash inflows process is defined as*

$$\Psi(t) = E\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)}\varphi(u)du|\mathcal{F}(t)\right], \tag{15}$$

where, $\Lambda(t) = \dfrac{Z(t)}{B(t)} = \exp(-rt)Z(t)$ *is the stochastic discount factor which adjusts for nominal interest rate and market price of risks, and* $E(.|\mathcal{F}(t))$ *is a real world conditional expectation with respect to the Brownian filtration* $(\mathcal{F}(t))_{t\geq 0}$. *For detail on real world measure* $\mathbb{P}$, *see Zhang [6], Nkeki [7], Nkeki and Nwozo [8].*

Proposition 1. *Let* $\Psi(t)$ *be the expected value of the discounted future cash inflows (EVDFCI) process, then*

$$\Psi(t) = \frac{\varphi(t)}{\phi}(\exp[\phi(T-t)] - 1), \tag{16}$$

where $\phi = \omega - r - \theta^\varphi \cdot \theta$.

Proof. By definition,

$$\begin{aligned}\Psi(t) &= E\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)}\varphi(u)du|\mathcal{F}(t)\right] \\ &= \varphi(t)E\left[\int_t^T \frac{\Lambda(u)\varphi(u)}{\Lambda(t)\varphi(t)}du|\mathcal{F}(t)\right].\end{aligned}$$

But, the processes $\Lambda(.)$ and $\varphi(.)$ are geometric Brownian motions and it follows that $\frac{\Lambda(u)\varphi(u)}{\Lambda(t)\varphi(t)}$ is independent of the Brownian filtration $\mathcal{F}(t)$, $u \geq t$. Hence,

$$\begin{aligned}\Psi(t) &= \varphi(t)E\left[\int_0^{T-t} \Lambda(s)\frac{\varphi(s)}{\varphi(0)}ds\right] \\ &= \varphi(t)E\left[\int_0^{T-t} \exp(-rs)Z(s)\exp((\omega - \frac{1}{2}\|\sigma^{\varphi}\|^2)s + \sigma^{\varphi\prime}W(s))ds\right] \\ &= \varphi(t)E\left[\int_0^{T-t} \exp(-rs - \frac{1}{2}\|\theta\|^2 s - \theta' W(s)) + (\omega - \frac{1}{2}\|\sigma^{\varphi}\|^2)s + \sigma^{\varphi\prime}W(s))ds\right] \\ &= \varphi(t)E\left[\int_0^{T-t} \exp((\omega - r)s\exp(-\frac{1}{2}(\|\theta\|^2 + \|\sigma^{\varphi}\|^2)s + (\sigma^{\varphi} - \theta)'W(s)ds\right] \\ &= \varphi(t)E\left[\int_0^{T-t} \exp((\omega - r - \sigma^{\varphi}\cdot\theta)s\exp(-\frac{1}{2}(\|\sigma^{\varphi} - \theta\|^2 s + (\sigma^{\varphi} - \theta)'W(s)ds\right] \\ &= \varphi(t)\left[\int_0^{T-t} \exp(\phi s)ds\right]\end{aligned}$$

where, $\phi = \omega - r - \sigma^{\varphi}\cdot\theta$.

Therefore,

$$\Psi(t) = \frac{\varphi(t)}{\phi}\left[\exp(\phi(T-t) - 1\right]. \tag{17}$$

□

The present value of the discounted future cash inflows (PVDFCI) is obtain by setting $t = 0$ in (17) and is given by

$$\Psi(0) = \frac{\varphi_0}{\phi}[\exp(\phi T) - 1]. \tag{18}$$

Proposition 1 tells us that the value of expected future cash inflows process $\Psi(t)$ is proportional to the instantaneous total cash inflows process $\varphi(t)$. Observe that at time T, the value of the inflow of cash is zero. This is because the value Ψ_0 has been invested while setting up the investment.

Taking the differential of both sides of (17), we obtain

$$d\Psi(t) = \Psi(t)[(r + \sigma_1^{\varphi}\theta^I + \sigma_2^{\varphi}\theta^S)dt + \sigma_1^{\varphi}dW^I(t) + \sigma_2^{\varphi}dW^S(t)] - \varphi(t)dt. \tag{19}$$

4.2. The Expected Value of Discounted Cash Outflows

Definition 3. *The expected discounted flow of cash outflows process at time t is defined as*

$$\Phi(t) = E\left[\int_t^{T+t} \frac{\Lambda(u)}{\Lambda(t)}\Gamma(u)du|\mathcal{F}(t)\right], T \geq t. \tag{20}$$

The contingent claim $\Gamma(t)$ that matures at the stopping time $t \in [0, T]$ is an $\mathcal{F}(t)$-measurable non-negative payoff that possesses a finite expectation. As outlined in [7], the value $\Phi(t)$ (the cash outflows process) can be obtained at time t by the real-world pricing formula given in (20).

Proposition 2. *Let $\Phi(t)$ be the expected discounted cash outflows (EDCO) process, then*

$$\Phi(t) = \frac{\Gamma(t)}{\beta}\left(1 - \exp[-\beta T]\right), \tag{21}$$

where $\beta = \delta - r - \sigma^{\Gamma} \cdot \theta$.

Proof. By definition,

$$\begin{aligned}
\Phi(t) &= E\left[\int_t^{T+t} \frac{\Lambda(u)}{\Lambda(t)}\Gamma(u)du|\mathcal{F}(t)\right] \\
&= \Gamma(t)E\left[\int_t^{T+t} \frac{\Lambda(u)\Gamma(u)}{\Lambda(t)\Gamma(t)}du|\mathcal{F}(t)\right].
\end{aligned}$$

But, the processes $\Lambda(.)$ and $\Gamma(.)$ are geometric Brownian motions and it follows that $\frac{\Lambda(u)\Gamma(u)}{\Lambda(t)\Gamma(t)}$ is independent of the Brownian filtration $\mathcal{F}(t)$, $u \geq t$. Adopting change of variables, we have

$$\begin{aligned}
\Phi(t) &= \Gamma(t)E\left[\int_0^{T} \Lambda(\tau)\frac{\Gamma(\tau)}{\Gamma(0)}d\tau\right] \\
&= \Gamma(t)E\left[\int_0^{T} \exp(-r\tau)Z(\tau)\frac{\Gamma(\tau)}{\Gamma(0)}d\tau\right]
\end{aligned}$$

Using (10), we have

$$\Phi(t) = \Gamma(t) E \left[\int_0^T \exp(-r\tau) Z(\tau) \exp((\delta - \frac{1}{2}\|\sigma^\Gamma\|^2)\tau + \sigma^{\Gamma\prime} W(\tau)) d\tau \right] \tag{22}$$

Applying parallelogram law on (22), we have

$$\Phi(t) = \Gamma(t) E \left[\int_0^T \exp(\delta - r)\tau \exp((-\frac{1}{2}\|\sigma^\Gamma - \theta\|^2 - \sigma^\Gamma \cdot \theta)\tau + (\sigma^\Gamma - \theta)' W(\tau)) d\tau \right] \tag{23}$$

Simplifying, (23), we have

$$\begin{aligned} \Phi(t) &= \Gamma(t) E \left[\int_0^T \exp(\delta - r - \sigma^\Gamma \cdot \theta)\tau \exp((-\frac{1}{2}\|\sigma^\Gamma - \theta\|^2)\tau + (\sigma^\Gamma - \theta)' W(\tau)) d\tau \right] \\ &= \Gamma(t) \int_0^T \exp(\delta - r - \sigma^\Gamma \cdot \theta)\tau d\tau \\ &= \Gamma(t) \int_0^T \exp(\beta\tau) d\tau \end{aligned}$$

where $\beta = \delta - r - \sigma^\Gamma \cdot \theta$.

Therefore,

$$\Phi(t) = \frac{\Gamma(t)}{\beta}[1 - \exp(-\beta T)]. \tag{24}$$

□

The present value of discounted future cash outflows (PVDFCO) is obtain as

$$\Phi_0 = \frac{\Gamma_0}{\beta}[1 - \exp(-\beta T)]. \tag{25}$$

Remark 2. *(i) If we allow T to be very large and $\beta > 0$, we have that Φ_0 will tends to $\frac{\Gamma_0}{\beta}$ i.e., $\lim_{T \longrightarrow \infty} \Phi_0 = \frac{\Gamma_0}{\beta}$ provided that $\beta > 0$.*
(ii) If we allow β to be relatively small, then Φ_0 will be very large and vice versa.

Proposition 2 tells us that the expected discounted cash outflows process $\Phi(t)$ is proportional to the instantaneous total cash outflows process $\Gamma(t)$.

Lemma 1. *Let $\Phi(t)$ be the expected discounted cash outflows process, then*
$d\Phi(t) = \Phi(t)[\alpha dt + \sigma_1^\Gamma dW^I(t) + \sigma_2^\Gamma dW^S(t)] - \Gamma(t)dt$,
where $\alpha = \dfrac{\delta(1 - \exp(-\beta T)) + \beta}{1 - \exp(-\beta T)}$.

Proof. Taking the differential of both sides of (24), we obtain

$$\begin{aligned}
d\Phi(t) &= \left(\frac{1-\exp(-\beta T)}{\beta}\right) d\Gamma(t) \\
&= \left(\frac{1-\exp(-\beta T)}{\beta}\right) \Gamma(t) \left((\delta + \frac{\beta}{1-\exp(-\beta T)})dt + {\sigma_1}^{\Gamma} dW^I(t) + {\sigma_2}^{\Gamma} dW^S(t)\right) - \Gamma(t)dt \\
&= \left(\frac{1-\exp(-\beta T)}{\beta}\right) \Gamma(t) \left(\alpha dt + {\sigma_1}^{\Gamma} dW^I(t) + {\sigma_2}^{\Gamma} dW^S(t)\right) - \Gamma(t)dt
\end{aligned}$$

Therefore,

$$d\Phi(t) = \Phi(t)(\alpha dt + {\sigma_1}^{\Gamma} dW^I(t) + {\sigma_2}^{\Gamma} dW^S(t)) - \Gamma(t)dt, \tag{26}$$

where, $\alpha = \dfrac{\delta(1-\exp(-\beta T)) + \beta}{1-\exp(-\beta T)}$. □

Obviously, the dynamics of the cash outflows and cash inflows processes in this paper have similar features. The difference is that the formal is seen as a form of cash outflow to be received by the holder at the maturity date while the later is seen as a form of cash inflow that is invested optimally by the investor.

5. Wealth Valuation for the Investor

This section consider the valuation of the investor's wealth at time t.

Definition 4. *The value of wealth process of the investor at time t is define as*

$$V(t) = X^{\varphi\Gamma}(t) + \Psi(t) - \Phi(t). \tag{27}$$

The value of wealth, $V(t)$ equals the wealth, $X^{\varphi\Gamma}(t)$ plus the discounted expected value of future cash inflows, $\Psi(t)$ less the discounted expected value of cash outflows, $\Phi(t)$.

Proposition 3. *The change in wealth of the investor is given by the dynamics*

$$dV(t) = (X^{\varphi\Gamma}(t), \Psi(t), \Phi(t)) \begin{pmatrix} (r + \Delta(t)A)dt + (\Sigma\Delta'(t))'dW(t) \\ (r + \sigma_\varphi \cdot \theta)dt + \sigma_\varphi dW(t) \\ -\alpha dt - \sigma_\Gamma dW(t) \end{pmatrix} \tag{28}$$

Equivalently,

$$dV(t) = V(t)[(r1_3 + D(t))dt + E(t)dW(t)], V(0) = v_0, \tag{29}$$

where, $D(t) = (\Delta(t)A, \sigma_\varphi \cdot \theta, -r - \alpha)'$; $E(t) = ((\Sigma\Delta'(t))', \sigma_\varphi, -\sigma_\Gamma)'$, *where* 1_3 *is a unit vector.*

Proof. Taking the differential of both sides of (27) and substituting in (11) and (19), the result follows. □

In the next section, we present the optimal portfolio strategies for the investor.

6. Optimal Portfolio as a Function of Cash Inflows and Cash Outflows

In this section, we consider the optimal portfolio process for the investor that depend on the cash flows processes. We define the general value function

$$J_1(t, v) = E[u(V(T))|V(t) = v]$$

where $J_1(t, v)$ is the path of $V(t)$ given the portfolio strategy $\Delta(t) = (\Delta^I(t), \Delta^S(t))$. Define $A(v)$ to be the set of all admissible portfolio strategy that are $\mathcal{F}_v$-progressively measurable, and let $U(t, v)$ be a concave function in $V(t)$ such that

$$U(t, v) = \sup_{\Delta \in A(v)} E[U(V(T))|V(t) = v] = \sup_{\Delta \in A(v)} J_1(t, v) \tag{30}$$

Then $U(t, v)$ satisfies the HJB equation

$$\begin{gathered} U_t + \sup\nolimits_{\Delta \in A(v)} \mathcal{HV}(t, v) = 0, \\ \text{subject to:} U(T, v) = \frac{1}{\gamma} v^\gamma. \end{gathered} \tag{31}$$

where,

$$\begin{aligned} \mathcal{HV}(t, v) &= rxU_x + r\Psi U_\Psi - \alpha\Phi U_\Phi + \Delta(t)AxU_x + \Psi\sigma_\varphi \cdot \theta + \frac{1}{2}\Sigma\Delta'(t)\Delta(t)\Sigma' x^U_{xx} \\ &+ x\Psi\Sigma'\Delta(t)\sigma'_\varphi U_{x\Psi} - x\Phi\Sigma'\Delta(t)\sigma'_\Gamma U_{x\Phi} + \frac{1}{2}\Psi^2\sigma_\varphi\sigma'_\varphi U_{\Psi\Psi} - \frac{1}{2}\Psi\Phi\sigma_\varphi\sigma'_\Gamma U_{\Psi\Phi} \\ &- \frac{1}{2}\Psi\Phi\sigma_\Gamma\sigma'_\varphi U_{\Phi\Psi} + \frac{1}{2}\Phi^2\sigma_\Gamma\sigma'_\Gamma U_{\Phi\Phi}. \end{aligned} \tag{32}$$

Since U is a concave function in $V(t)$ and $U(t,v) \in C^{1,2}(R \times [0,T])$, then (31) is well defined. We now find the partial derivative of $\mathcal{HV}(t,v)$ with respect to $\Delta(t)$ and set to zero as follows:

$$\frac{\mathcal{HV}(t,v)}{\Delta(t)} = AxU_x + \Sigma\Delta'(t)\Sigma' x^2 U_{xx} + x\Psi\Sigma'\sigma'_\varphi - x\Phi\Sigma'\sigma'_\Gamma = 0. \tag{33}$$

From (33), we have

$$(\Delta'(t))^* = -\frac{(\Sigma\Sigma')^{-1}A}{x}\frac{U_x}{U_{xx}} - \frac{\Psi\Sigma^{-1}\sigma'_\varphi U_{x\Psi}}{xU_{xx}} + \frac{\Phi\Sigma^{-1}\sigma'_\Gamma U_{x\Phi}}{xU_{xx}}. \tag{34}$$

This is the variational form of Merton portfolio value, (see Nkeki and Nwozo 2012).

(i) We find that it is optimal to invest in the portfolio $-\frac{(\Sigma\Sigma')^{-1}A}{x}\frac{U_x}{U_{xx}}$ proportional to the market price of risk corresponding to stock and inflation-linked bond through the relative risk aversion index. Note that

$$-\frac{U_x}{U_{xx}}$$

represents part of the portfolio that depend on individual risk preferences, wealth process, expected value discounted cash inflows and expected value discounted cash outflows.

(ii) an hedging portfolio $-\frac{\Psi\Sigma^{-1}\sigma'_\varphi U_{x\Psi}}{xU_{xx}}$ is proportional to the stochastic cash inflows through the cross derivative of the value function U,

where

$$-\frac{U_{x\Psi}}{U_{xx}}$$

represents part of the portfolio that depend on individual risk preferences.

(iii) an hedging portfolio $\frac{\Phi\Sigma^{-1}\sigma'_\Gamma U_{x\Phi}}{xU_{xx}}$ is proportional to the stochastic cash outflows through the cross derivative of the value function U,

where

$$\frac{U_{x\Phi}}{U_{xx}}$$

represents part of the portfolio that depend on individual risk preferences.

Note that $\frac{(\Sigma\Sigma')^{-1}A}{x}$, $\frac{\Psi\Sigma^{-1}\sigma'_\varphi}{x}$ and $\frac{\Phi\Sigma^{-1}\sigma'_\Gamma}{x}$ are three components that depend only on the financial market structure. It implies that the investment policies corresponding to the three components do not require the knowledge of the preferences or the endowments of the funds of the investor (see Battocchio, [10]). (ii) and (iii) can be seen as inter-temporal hedging terms that offset any shock to the stochastic cash inflows and outflows.

In the next section, we consider the value of wealth of the investor.

7. The Value of Optimal Wealth and Cash Flows Dependent on the Optimal Wealth of the Investor

In this section, we consider the value of wealth and cash inflows and cash outflows that depends on the optimal wealth of the investor. We now determine the following:

$$\Delta(t)A = -\frac{A'[(\Sigma\Sigma')^{-1}]'AU_x}{xU_{xx}} - \frac{\Psi[\Sigma^{-1}\sigma'_\varphi]'AU_{x\Psi}}{xU_{xx}} + \frac{\Phi[\Sigma^{-1}\sigma'_\Gamma]'AU_{x\Phi}}{xU_{xx}}$$

But, $\Sigma\Sigma^{-1} = \mathbf{1}$, which is a 2×2 identity matrix.

$$(\Sigma\Delta'(t))' = -\frac{[(\Sigma')^{-1}A]'U_x}{xU_{xx}} - \frac{\Psi\sigma_\varphi U_{x\Psi}}{xU_{xx}} + \frac{\Phi\sigma_\Gamma U_{x\Phi}}{xU_{xx}}$$

We then determine $D(t)$ and $E(t)$ as follow:.

$$D(t) = \left(-\frac{A'[(\Sigma\Sigma')^{-1}]'AU_x}{xU_{xx}} - \frac{\Psi[\Sigma^{-1}\sigma'_\varphi]'AU_{x\Psi}}{xU_{xx}} + \frac{\Phi[\Sigma^{-1}\sigma'_\Gamma]'AU_{x\Phi}}{xU_{xx}}, \sigma_1^\varphi\theta^I + \sigma_2^\varphi\theta^S, -r-\alpha\right)';$$

$$E(t) = \left(-\frac{[(\Sigma')^{-1}A]'U_x}{xU_{xx}} - \frac{\Psi\sigma_\varphi U_{x\Psi}}{xU_{xx}} + \frac{\Phi\sigma_\Gamma U_{x\Phi}}{xU_{xx}}, \sigma_\varphi, \sigma_\Gamma\right)';$$

But, $\dfrac{U_x}{U_{xx}} = \dfrac{1}{\gamma-1}(x+\Psi-\Phi)$, $\dfrac{U_{x\Psi}}{U_{xx}} = 1$ and $\dfrac{U_{x\Phi}}{U_{xx}} = -1$.

$$D(t) = \left(\frac{A'[(\Sigma\Sigma')^{-1}]'A(x+\Psi-\Phi)}{(1-\gamma)x} - \frac{\Psi[\Sigma^{-1}\sigma'_\varphi]'A}{x} - \frac{\Phi[\Sigma^{-1}\sigma'_\Gamma]'A}{x}, \sigma_1^\varphi\theta^I + \sigma_2^\varphi\theta^S, -r-\alpha\right)';$$

$$E(t) = \left(\frac{[\mathbf{1}(\Sigma')^{-1}A]'(x+\Psi-\Phi)}{(1-\gamma)x} - \frac{\Psi\mathbf{1}\sigma_\varphi}{x} - \frac{\Phi\mathbf{1}\sigma_\Gamma}{x}, \sigma_\varphi, \sigma_\Gamma\right)'.$$

The wealth process of the investor is obtain as follows:

(35)

$$dX^{\varphi\Gamma*}(t) = (rX^{\varphi\Gamma*}(t) + \frac{A'[(\Sigma\Sigma')^{-1}]'AX^{\varphi\Gamma*}(t)}{1-\gamma} + \frac{A'[(\Sigma\Sigma')^{-1}]'A(\Psi(t)-\Phi(t))}{1-\gamma}$$
$$-\Psi(t)[\Sigma^{-1}\sigma'_\varphi]'A - \Phi(t)[\Sigma^{-1}\sigma'_\Gamma]'A)dt + (\frac{((\Sigma')^{-1}A)'X^{\varphi\Gamma*}(t)}{1-\gamma} + \frac{((\Sigma')^{-1}A)'(\Psi(t)-\Phi(t)))}{1-\gamma}$$
$$-\Psi(t)\sigma_\varphi - \Phi(t)\sigma_\Gamma)dW(t)$$

It follows that

$$
\begin{aligned}
dV^*(t) &= (rX^*(t) + \frac{A'[(\Sigma\Sigma')^{-1}]'AX^*(t)}{1-\gamma})dt + \frac{((\Sigma')^{-1}A)'X^*(t)}{1-\gamma}dW(t)\\
&+\Psi(t)((r+\sigma_\varphi\cdot\theta + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} - [\Sigma^{-1}\sigma'_\varphi]'A)dt + (\frac{[(\Sigma')^{-1}A]'}{1-\gamma})dW(t))\\
&-\Phi(t)((\alpha + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} + [\Sigma^{-1}\sigma'_\Gamma]'A)dt + (\frac{((\Sigma')^{-1}A)'}{1-\gamma} + 2\sigma_\Gamma)dW(t).
\end{aligned}
\tag{36}
$$

We therefore deduce that

$$
dX^*(t) = (rX^*(t) + \frac{A'[(\Sigma\Sigma')^{-1}]'AX^*(t)}{1-\gamma})dt + \frac{((\Sigma')^{-1}A)'X^*(t)}{1-\gamma}dW(t);
\tag{37}
$$

$$
d\Psi(t) = \Psi(t)((r+\sigma_\varphi\cdot\theta + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} - [\Sigma^{-1}\sigma'_\varphi]'A)dt + (\frac{[(\Sigma')^{-1}A]'}{1-\gamma})dW(t));
\tag{38}
$$

$$
d\Phi(t) = \Phi(t)((\alpha + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} + [\Sigma^{-1}\sigma'_\Gamma]'A)dt + (\frac{((\Sigma')^{-1}A)'}{1-\gamma} + 2\sigma_\Gamma)dW(t).
\tag{39}
$$

Therefore,

(40)

$$
\begin{aligned}
&V^*(t) = X^*(0)\exp[\int_0^t (r + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} - \frac{1}{2}\left(\frac{((\Sigma')^{-1}A)'((\Sigma')^{-1}A)}{(1-\gamma)^2}\right))ds\\
&+\int_0^t \frac{((\Sigma')^{-1}A)'}{1-\gamma}dW(s)] + \Psi(0)\exp[\int_0^t ((r+\sigma_\varphi\cdot\theta + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} - [\Sigma^{-1}\sigma'_\varphi]'A\\
&-\frac{1}{2}\left(\frac{((\Sigma')^{-1}A)'((\Sigma')^{-1}A)}{(1-\gamma)^2}\right))ds + \int_0^t \frac{[(\Sigma')^{-1}A]'}{1-\gamma}dW(s))]\\
&-\Phi(0)\exp[\int_0^t ((\alpha + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} + [\Sigma^{-1}\sigma'_\Gamma]'A\\
&-\frac{1}{2}\left(\frac{((\Sigma')^{-1}A)'(\Sigma')^{-1}A)}{(1-\gamma)^2} + 2\frac{((\Sigma')^{-1}A)'\sigma'_\Gamma}{1-\gamma} + 4\sigma_\Gamma\sigma'_\Gamma)\right))ds + \int_0^t (\frac{((\Sigma')^{-1}A)'}{1-\gamma} + 2\sigma_\Gamma)dW(s)].
\end{aligned}
$$

The terminal value of the investor's optimal wealth is obtain as follows

(41)

$$
\begin{aligned}
&V^*(T) = X^*(0)\exp[\int_0^T (r + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} - \frac{1}{2}\left(\frac{((\Sigma')^{-1}A)'((\Sigma')^{-1}A)}{(1-\gamma)^2}\right))ds\\
&+\int_0^T \frac{((\Sigma')^{-1}A)'}{1-\gamma}dW(s)] + \Psi(0)\exp[\int_0^T ((r+\sigma_\varphi\cdot\theta + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} - [\Sigma^{-1}\sigma'_\varphi]'A\\
&-\frac{1}{2}\left(\frac{((\Sigma')^{-1}A)'((\Sigma')^{-1}A)}{(1-\gamma)^2}\right))ds + \int_0^T \frac{[(\Sigma')^{-1}A]'}{1-\gamma}dW(s))]\\
&-\Phi(0)\exp[\int_0^T ((\alpha + \frac{A'[(\Sigma\Sigma')^{-1}]'A}{1-\gamma} + [\Sigma^{-1}\sigma'_\Gamma]'A\\
&-\frac{1}{2}\left(\frac{((\Sigma')^{-1}A)'(\Sigma')^{-1}A)}{(1-\gamma)^2} + 2\frac{((\Sigma')^{-1}A)'\sigma'_\Gamma}{1-\gamma} + 4\sigma_\Gamma\sigma'_\Gamma)\right))ds + \int_0^T (\frac{((\Sigma')^{-1}A)'}{1-\gamma} + 2\sigma_\Gamma)dW(s)].
\end{aligned}
$$

Observe from (40) that $\Psi(t)$ and $\Phi(t)$ can be re-express in terms of $X^*(t)$ as follows:

$$\Psi(t) = \frac{\Psi(0)X^*(t)}{X^*(0)} \exp\left[\left(\sigma_\varphi \cdot \theta - [\Sigma^{-1}\sigma_\varphi']'A\right) t\right]. \tag{42}$$

$$\Phi(t) = \frac{\Phi(0)X^*(t)}{X^*(0)} \exp\left[\left(\alpha - r + [\Sigma^{-1}\sigma_\Gamma']'A - (\frac{((\Sigma')^{-1}A)'\sigma_\Gamma'}{1-\gamma} + 2\sigma_\Gamma\sigma_\Gamma')\right) t + 2\sigma_\Gamma W(t)\right]. \tag{43}$$

These show that the values of the cash inflows and the outflows ultimately depend on the wealth process and the investor's risk preference over time. Now, equating (42) and (17), we have

$$\varphi(t) = \frac{\varphi_0(\exp(\phi T) - 1)X^*(t)}{X^*(0)(\exp(\phi(T-t)) - 1)} \exp\left[\left(\sigma_\varphi \cdot \theta - [\Sigma^{-1}\sigma_\varphi']'A\right) t\right], T > t. \tag{44}$$

Equating (43) and (24), we have

$$\Gamma(t) = \frac{\Gamma_0 X^*(t)}{X^*(0)} \exp\left[\left((\alpha - r + [\Sigma^{-1}\sigma_\Gamma']'A - (\frac{((\Sigma')^{-1}A)'\sigma_\Gamma'}{1-\gamma} + 2\sigma_\Gamma\sigma_\Gamma')\right) t + 2\sigma_\Gamma W(t)\right]. \tag{45}$$

From (44), we found that the cash inflows depend on the optimal wealth of the investor and the optimal wealth depend on the financial market. We therefore conclude that the behaviour of the financial market goes a long way in influencing the investor's cash inflows. When the market react positively, the investor is encourage to invest more and vice versa. Interesting, we found that risk associated with the cash inflows only comes from the financial market. This makes sense, since the outcome of the financial market affect the choice of making cash inflows by the investor. Observe that if we allow the volatilities of the cash inflows to be zero vector (that is, deterministic cash inflows), then (44) will becomes

$$\varphi(t) = \frac{\varphi_0(\exp(\phi T) - 1)X^*(t)}{X^*(0)(\exp(\phi(T-t)) - 1)}, \phi = \omega - r, T > t. \tag{46}$$

This shows that cash inflows depends on the stochastic wealth process and is therefore, stochastic. From (46), we found that the cash outflows also depend on the optimal wealth of the investor. This is an intuitive result since what should be payout will depend on the wealth generated from the investment. We also found that the cash outflows is exposed to financial market risks as well as cash outflow risks. Observe also that, if we allow the volatilities of the cash outflows to be zero vector (that is, deterministic cash outflows), then (45) will becomes

$$\Gamma(t) = \frac{\Gamma_0 X^*(t)}{X^*(0)} \exp\left[(\alpha - r)t\right], r = \delta - \beta. \tag{47}$$

This shows that cash outflows depend on the stochastic wealth process and the expected growth rate of the cash outflows, and is also stochastic.

8. Conclusion

The paper examined the optimal portfolios with stochastic cash inflows and outflows and expected terminal wealth for an investor. The portfolio values and expected terminal wealth that accrued to the investor were obtained. We found that it is optimal to invest in the portfolio which is proportional to the market price of risk corresponding to stock and inflation-linked bond through the relative risk aversion index. It was also found that the portfolio values of the investor involve inter-temporal hedging terms that offset any shocks to the stochastic cash inflows and outflows. It was further found that the cash inflows and cash outflows depend on the optimal wealth of the investor. The cash inflows was found to depends only on the risks associated with the financial market while the cash outflows depends on its own risks and the risks associated with the financial market.

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