Available online at http://scik.org

Mathematical Finance Letters, 2013, 2013:1

ISSN 2051-2929

OPTION PRICING IN THE MULTIDIMENSIONAL BLACK-SCHOLES MARKET WITH VASICEK INTEREST RATES

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Abstract: An explicit state-price deflator for the multidimensional Black-Scholes market with Vasicek stochastic interest rates is constructed. It is applied to obtain extensions of the Margrabe and Black-Scholes option pricing formulas. These formulas, which are validated in a multiple risk economy with stochastic interest rates, remain invariant under changing market prices of risk. Some comments including related research round up the exposé.

Keywords: state-price deflator; Black-Scholes; Margrabe; Vasicek interest rates.

2000 AMS Subject Classification: 60G15; 62P05; 91B24; 91G20; 91G80

1. Introduction

The concept of state-price deflator or stochastic discount factor, which has been introduced by Duffie (1992), is a convenient tool in general asset pricing theory. It contains information about the valuation of payments in different states at different points in time. The state-price deflator is a natural extension of the notion of state prices that were introduced earlier and studied by Arrow (1951/53/71), Debreu (1954), Negishi (1960) and Ross (1978), a milestone in the history of asset pricing (see Dimson and Mussavian (1999)). Though general frameworks for deriving state-price

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This paper is dedicated to the 40th anniversary of the Black-Scholes formula

Received December 4, 2012

deflators exist (e.g. Milterssen and Persson (1999) and Jeanblanc et al. (2009)), there are not many known explicit expressions for them and their distribution functions.

Recently, the author has introduced a multivariate Black-Scholes-Vasicek (BSV) deflator and has applied it to option pricing in [24]-[26]. Our next and present goal is an extension of the Black-Scholes deflator to a more general version with Vasicek interest rates as additional source of randomness. We obtain extensions of the Margrabe and Black-Scholes option pricing formulas and validate them in a multiple risk economy with Vasicek interest rates. The invariance of these formulas against changing market prices of risk, which has been first noticed in [24], is preserved in the extended model.

A short account of the content follows. Section 2 constructs the Gaussian state-price deflator for the multidimensional Black-Scholes market with Vasicek interest rates. Then, we show in Section 3 that Margrabe's formula remains valid in this multiple risk economy with stochastic interest rates. The core of our contribution is Section 4, which contains a new elementary approach to the generalized Black-Scholes formula with Vasicek interest rates. After parameter transformation it coincides with the formula by Kim (2002) that has been obtained with the equivalent martingale measure method. This earlier derivation applies equally simple algebra and the Kunitomo and Takahashi (1992) Lemma. We conclude with some comments including related research in this area.

2. A Gaussian state-price deflator in a multiple risk economy with stochastic interest rates

Consider a multiple risk economy with $m \geq 1$ risky assets, whose real-world prices with time horizon T satisfy the stochastic differential equations of Itô type

$$dS_t^{(k)} / S_t^{(k)} = \mu_k(t)dt + \sigma_k dW_t^{(k)}, \quad t \in [0,T], \quad k = 1,...,m, \tag{2.1}$$

where the σ_k 's are constant volatilities, the $\mu_k(t)$'s are arbitrary time dependent Gaussian drifts, and the $W_t^{(k)}$'s are correlated standard Wiener processes such that $E[dW_t^{(i)} dW_t^{(j)}] = \rho_{ij} dt$. The geometric Brownian motions (2.1) constitute a so-called multidimensional Black-Scholes market. The economy contains also an exogenously given money market account, whose value follows the real-world dynamics

$$dM_t / M_t = r(t)dt, \quad t \in [0,T], \tag{2.2}$$

where the short rate process follows a one-factor diffusion process of Vasicek (1977) type

$$dr(t) = \kappa\big(\theta - r(t)\big) \cdot dt + \sigma \cdot dW_t^{(m+1)}, \tag{2.3}$$

with $W_t^{(m+1)}$ a standard Wiener process that is correlated with the Wiener processes driving the risky assets such that $E[dW_t^{(k)} dW_t^{(m+1)}] = \rho_{k,m+1} dt,\ k = 1,...,m$. Let B_t^T be the price at time t of a zero-coupon bond paying 1 unit of account with certainty at maturity date T . According to Munk (2011), equations (7.14)-(7.15), the bond prices follow the dynamics

$$dB_t^T / B_t^T = \mu_B^T(t, r(t)) \cdot dt + \sigma_B^T(t, r(t)) \cdot dW_t^{(m+1)}, \tag{2.4}$$

where, for the Vasicek interest rate specification, the parametric functions are given by

$$\begin{aligned} &\mu_B^T(t, r(t)) = r(t) - \lambda_{m+1} \cdot \sigma_{m+1}(T-t), \\ &\sigma_{m+1}(T-t) = -\sigma_B^T(t, r(t)) = \kappa^{-1}(1 - e^{-\kappa(T-t)}) \cdot \sigma. \end{aligned} \tag{2.5}$$

The constant $\lambda_{m+1} < 0$ is the market price of the risky zero-coupon bond (the $m+1$-th asset in the economy). Note that a constant market price of risk can be justified using market equilibrium theory (e.g. Munk (2011), Section 5.4.2). One observes that the second term in (2.4) is negative (a positive shock to the short rate implies a negative shock to the zero-coupon bond price and vice versa). Therefore, the volatility of the bond price is actually the absolute value of $\sigma_B^T(t, r(t))$, which justifies herewith the definition of the time to maturity dependent volatility function $\sigma_{m+1}(T-t)$. Since in equilibrium risky assets have usually an expected rate of return that exceeds the instantaneous risk-free rate, the constant λ_{m+1} must be negative. Consider further the (constant) market prices $\lambda_k > 0$ of the first m risky assets defined by

$$\lambda_k \sigma_k = \mu_k(t) - r(t), \quad k = 1, ..., m. \tag{2.6}$$

A straightforward application of Itô's Lemma to the system of stochastic differential equations (2.1) and (2.4) (taking into account (2.5) and (2.6)) implies the following representations in terms of the integrated short rate process $R_t = \int_0^t r(s)ds$. For all $t \in [0, T]$ one has

$$\begin{aligned} S_t^{(k)} &= S_0^{(k)} \cdot \exp\left\{R_t + (\lambda_k \sigma_k - \tfrac{1}{2}\sigma_k^2) \cdot t + \sigma_k \cdot W_t^{(k)}\right\}, \quad k = 1, ..., m, \\ B_t^T &= B_0^T \cdot \exp\left\{\begin{array}{l} R_t - \lambda_{m+1} \cdot \int_0^t \sigma_{m+1}(T-s)ds \\ -\tfrac{1}{2}\int_0^t \sigma_{m+1}^2(T-s)ds - \int_0^t \sigma_{m+1}(T-s)dW_s^{(m+1)} \end{array}\right\}. \end{aligned} \tag{2.7}$$

Since the Vasicek short rate $r(t)$ (given the initial value $r(0)$) is normally distributed, the integrated short rate R_t is also normally distributed. It follows that the $m+1$ risky assets in (2.7) are exponential Gaussian processes with lognormal distributions. The state-price deflator in [24] generalizes as follows to the context of a multiple risk economy with stochastic interest rates.

Theorem 2.1. (Exponential Gaussian state-price deflator of dimension $m+1$) *Given is a Black-Scholes market with* $m \geq 1$ *risky assets in a stochastic Vasicek interest rate environment. Assume that the risky assets and the bond price follow the log-normal real-world prices* (2.7), *where the correlation matrix* $C = (\rho_{ij}), 1 \leq i, j \leq m+1$ *of the multivariate Wiener process* $W_t = (W_t^{(1)}, ..., W_t^{(m+1)})$ *is non-singular and positive semi-definite. Then, the exponential Gaussian process*

$$\begin{aligned} D_t^{(m+1)} &= \exp(-R_t - \tfrac{1}{2}\beta^T C\beta \cdot t - \beta^T W_t), \\ \beta &= C^{-1}\lambda, \quad \lambda = (\lambda_1, ..., \lambda_{m+1})^T, \quad t \in [0, T], \end{aligned} \tag{2.8}$$

is a well-defined state-price deflator.

Proof. According to the general theory of state-price deflators (e.g. Munk (2011), Section 4.3), the stochastic process (2.8) defines a deflator provided the following conditions are fulfilled:

$$\begin{aligned} &D1. \quad D_0^{(m+1)} = 1, \quad D_t^{(m+1)} > 0, \quad \forall t \in [0,T], \quad all\ states, \quad Var[D_t^{(m+1)}] < \infty, \quad \forall t \in [0,T], \\ &D2. \quad E[D_t^{(m+1)} \exp(R_t)] = 1, \quad \forall t \in [0,T], \\ &D3. \quad E[D_t^{(m+1)} S_t^{(k)}] = S_0^{(k)}, k = 1,...,m, \quad E[D_t^{(m+1)} B_t^T] = B_0^T, \quad \forall t \in [0,T]. \end{aligned}$$

The first condition D1 is trivially fulfilled. The conditions D2 and D3 mean that the discounted cumulative interest rate process and the discounted risky asset prices are martingales. The validity of D2 follows from the fact that $\ln\{D_t^{(m+1)} \exp(R_t)\} = -\frac{1}{2}\beta^T C\beta \cdot t - \beta^T W_t$ is normally distributed with mean $-\frac{1}{2}\beta^T C\beta \cdot t$ and variance $\beta^T C\beta \cdot t$. To show the first part of D3, let $m_k(t)$ and $v_k^2(t)$ be the mean and variance of the normally distributed random variable $X_t^{(k)} = \ln\{D_t^{(m+1)} S_t^{(k)}\} = \ln S_0^{(k)} - \frac{1}{2}\beta^T C\beta \cdot t - \beta^T W_t + (\lambda_k \sigma_k - \frac{1}{2}\sigma_k^2) \cdot t + \sigma_k \cdot W_t^{(k)}$. Then, one has

$$m_k(t) = \ln S_0^{(k)} - \tfrac{1}{2}\beta^T C\beta \cdot t + (\lambda_k \sigma_k - \tfrac{1}{2}\sigma_k^2) \cdot t, \quad k = 1,...,m,$$

$$v_k^2(t) = \beta^T C\beta \cdot t + \sigma_k^2 t - 2 \cdot \sum_j \rho_{kj}\beta_j \sigma_k t = \beta^T C\beta \cdot t + (\sigma_k^2 - 2\lambda_k \sigma_k) \cdot t,$$

where the last equality follows from the fact that $C\beta = \lambda$. It follows that

$$E[D_t^{(m+1)} S_t^{(k)}] = E[\exp(X_t^{(k)})] = \exp\left\{m_k(t) + \tfrac{1}{2} v_k^2(t)\right\} = S_0^{(k)},\ k = 1,...,m.$$

Similarly, let $m_B(t)$ and $v_B^2(t)$ be the mean and variance of the normal random variable

$$Y_t = \ln\left\{D_t^{(m+1)} B_t^T\right\} = \ln B_0^T - \tfrac{1}{2}\beta^T C\beta \cdot t - \beta^T W_t - \lambda_{m+1} \cdot \int_0^t \sigma_{m+1}(T-s)ds$$
$$-\tfrac{1}{2}\int_0^t \sigma_{m+1}^2(T-s)ds - \int_0^t \sigma_{m+1}(T-s)dW_s^{(m+1)}.$$

From the well-known rules of stochastic calculus (e.g. Munk (2011), Chapter 3), one gets

$$m_B(t) = \ln B_0^T - \tfrac{1}{2}\beta^T C\beta \cdot t - \lambda_{m+1} \cdot \int_0^t \sigma_{m+1}(T-s)ds - \tfrac{1}{2}\int_0^t \sigma_{m+1}^2(T-s)ds\ ,$$

$$v_B^2(t) = \beta^T C\beta \cdot t + \int_0^t \sigma_{m+1}^2(T-s)ds + 2\cdot\sum_j \rho_{j,m+1}\beta_j \cdot \int_0^t \sigma_{m+1}(T-s)ds$$
$$= \beta^T C\beta \cdot t + \int_0^t \sigma_{m+1}^2(T-s)ds + 2\lambda_{m+1} \cdot \int_0^t \sigma_{m+1}(T-s)ds,$$

where again the equation $C\beta = \lambda$ has been used. It follows that

$$E[D_t^{(m+1)} B_t^T] = E[\exp(Y_t)] = \exp\left\{m_B(t) + \tfrac{1}{2} v_B^2(t)\right\} = B_0^T .$$

This shows D3 and completes the proof. ◊

3. Margrabe's formula in a stochastic interest rate environment

The exchange option pricing formula by Margrabe (1978) is validated in the multiple risk economy with Vasicek interest rates. To show this, one needs a further elementary result from probability theory. Suppose that the random vector (S_1, S_2) has a bivariate lognormal distribution with parameter vector $(\mu_1, \upsilon_1, \mu_2, \upsilon_2, \rho)$ such that the random vector

$$(U_1, U_2) = (\{\ln S_1 - \mu_1\}/\upsilon_1, \{\ln S_2 - \mu_2\}/\upsilon_2) \tag{3.1}$$

has a standard bivariate normal distribution with correlation coefficient ρ.

Lemma 3.1. *Let* $\Phi(x)$ *be the standard normal distribution. Then the expected spread reads*

$$\begin{aligned} E[(S_1 - S_2)_+] &= \exp\left(\mu_1 + \tfrac{1}{2}\upsilon_1^2\right)\Phi\left(\frac{\mu_1 - \mu_2 + \upsilon_1^2 - \rho\upsilon_1\upsilon_2}{\sqrt{\upsilon_1^2 + \upsilon_2^2 - 2\rho\upsilon_1\upsilon_2}}\right) \\ &- \exp\left(\mu_2 + \tfrac{1}{2}\nu_2^2\right)\Phi\left(\frac{\mu_1 - \mu_2 - \upsilon_2^2 + \rho\upsilon_1\upsilon_2}{\sqrt{\upsilon_1^2 + \upsilon_2^2 - 2\rho\upsilon_1\upsilon_2}}\right). \end{aligned} \tag{3.2}$$

Theorem 3.1. (Margrabe for the multidimensional Black-Scholes market with Vasicek interest rates) *Under the assumptions of Theorem* 2.1, *the market value at initial time of a European exchange option on the risky assets with real-world prices* $S_t^{(k)}, S_t^{(\ell)}, k \neq \ell \in \{1, ..., m\}$ *and strike time* T *is given by the formula*

$$\begin{aligned} &E[D_T^{(m+1)}(S_T^{(k)} - S_T^{(\ell)})_+] \\ &= S_0^{(k)}\Phi\left(\frac{\ln\{S_0^{(k)}/S_0^{(\ell)}\} + \tfrac{1}{2}\upsilon^2 T}{\upsilon\sqrt{T}}\right) - S_0^{(\ell)}\Phi\left(\frac{\ln\{S_0^{(k)}/S_0^{(\ell)}\} - \tfrac{1}{2}\upsilon^2 T}{\upsilon\sqrt{T}}\right), \\ &\upsilon^2 = \sigma_k^2 + \sigma_\ell^2 - 2\rho_{k\ell}\sigma_k\sigma_\ell, \quad k \neq \ell \in \{1, ..., m\}. \end{aligned} \tag{3.3}$$

Proof. It suffices to show (3.3) for $k = 1, \ell = 2$. By assumption the random variables

$$\begin{aligned} X_k &= \ln\{D_T^{(m+1)} S_T^{(k)}\} \\ &= \ln S_0^{(k)} - \tfrac{1}{2}\beta^T C\beta \cdot T - \beta^T W_T + (\lambda_k\sigma_k - \tfrac{1}{2}\sigma_k^2) \cdot T + \sigma_k \cdot W_T^{(k)}, \quad k = 1,2, \end{aligned}$$

are normally distributed. Let $(\mu_1, \upsilon_1, \mu_2, \upsilon_2, \rho)$ denote their means, standard deviations and correlation coefficient. From the proof of Theorem 2.1 one borrows the identities

$$\mu_k = \ln S_0^{(k)} - \tfrac{1}{2}\beta^T C\beta \cdot T + (\lambda_k\sigma_k - \tfrac{1}{2}\sigma_k^2) \cdot T,$$

$$\upsilon_k^2 = \beta^T C\beta \cdot T + (\sigma_k^2 - 2\lambda_k\sigma_k) \cdot T, \quad k = 1,2,$$

which imply that $\mu_k + \frac{1}{2}\upsilon_k^2 = \ln\left\{S_0^{(k)}\right\}, \quad k = 1,2$. Furthermore, one notes that

$$X_1 - X_2 = \ln\left\{S_0^{(1)} / S_0^{(2)}\right\} + (\lambda_1\sigma_1 - \lambda_2\sigma_2 - \tfrac{1}{2}(\sigma_1^2 - \sigma_2^2)) \cdot T + \sigma_1 \cdot W_T^{(1)} - \sigma_2 \cdot W_T^{(2)},$$

from which one deduces that

$$\upsilon_1^2 + \upsilon_2^2 - 2\rho\upsilon_1\upsilon_2 = Var[X_1 - X_2] = (\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2) \cdot T = \upsilon^2 T .$$

Similarly, one has

$$\begin{aligned} \upsilon_1^2 - \rho\upsilon_1\upsilon_2 &= Cov[X_1, X_1 - X_2] = Cov[\sigma_1 \cdot W_T^{(1)} - \beta^T W_T, \sigma_1 \cdot W_T^{(1)} - \sigma_2 \cdot W_T^{(2)}] \\ &= (\sigma_1^2 - \rho_{12}\sigma_1\sigma_2 - \sum_j \rho_{1j}\beta_j\sigma_1 + \sum_j \rho_{2j}\beta_j\sigma_2) \cdot T = (\sigma_1^2 - \rho_{12}\sigma_1\sigma_2 - \lambda_1\sigma_1 + \lambda_2\sigma_2) \cdot T, \end{aligned}$$

where further use has been made of the identity $C\beta = \lambda$. From this and the above values for the means one gets without difficulty that

$$\begin{aligned} \mu_1 - \mu_2 + \upsilon_1^2 - \rho\upsilon_1\upsilon_2 &= \ln\left\{S_0^{(1)} / S_0^{(2)}\right\} + \tfrac{1}{2}\upsilon^2 T, \\ \mu_1 - \mu_2 - \upsilon_2^2 + \rho\upsilon_1\upsilon_2 &= \ln\left\{S_0^{(k)} / S_0^{(\ell)}\right\} - \tfrac{1}{2}\upsilon^2 T. \end{aligned}$$

Inserting the obtained expressions into (3.2) one obtains (3.3) for $k = 1, \ell = 2$. ◊

Remarks 3.1. Poulsen (2010), end of Section 1, has observed that Margrabe's formula is still valid with stochastic interest rates, provided the factors that drive interest rates are independent from those driving the risky assets. Bernard and Cui (2010) show that this is true even when interest rates and risky assets are non-trivially dependent. Its validity in a multiple risk economy with Vasicek interest rates is new as is its different novel and short elementary proof. Furthermore, the present result generalizes the previous one in [24], Theorem 2. One might challenge whether the multivariate maximum, minimum and maximum spread option formulas in [26] also generalize to the present and even more general contexts.

4. Black-Scholes formula in a stochastic interest rate environment

The original European call option formula by Black and Scholes (1973) (see also Merton (1973)) is extended to the multiple risk economy with Vasicek interest rates. The elementary derivation of the generalized formula is done along the line of Section 3, where some additional stochastic calculus formulas related to the integrated Vasicek short rate process are required. Let us start from the short rate dynamics under the risk-neutral probability measure, also called Q-measure, such that

$$dr(t) = \kappa\big(\theta - r(t)\big) \cdot dt + \sigma \cdot dW_t^Q . \tag{4.1}$$

According to general asset pricing theory (e.g. Munk (2011), Section 4.8, equation (4.25)), the dynamics under the real-world probability measure, also called P-measure, turns out to be

$$dr(t) = \kappa\big(\theta + \lambda_{m+1}\sigma\kappa^{-1} - r(t)\big) \cdot dt + \sigma \cdot dW_t^{(m+1)} . \tag{4.2}$$

Alternatively, some authors start from a P-measure of the type (4.1) and transform it to the corresponding Q-measure (see the later comments about this equivalent approach). The following formulas enter into the generalized Black-Scholes formula (4.4) below. The identity F1 is a crucial ingredient in the proof of Theorem 4.1.

Lemma 4.1. *Assume the real-world short rate process follows the Vasicek stochastic differential equation* (4.2) *and let* $R_T = \int_0^T r(s)ds$ *be the associated integrated short rate process at maturity date* T *. Then the following identities hold true:*

$$\begin{aligned}
&\text{F1.} \quad E[R_T] = E^Q[R_T] + \lambda_{m+1} \cdot Cov[R_T, W_T^{(m+1)}], \\
&\text{F2.} \quad E^Q[R_T] = \theta \cdot T + \{r(0) - \theta\} \cdot B(T), \quad Cov[R_T, W_T^{(m+1)}] = (\sigma/\kappa) \cdot \{T - B(T)\}, \\
&\text{F3.} \quad \mathrm{Var}[R_T] = (\sigma/\kappa)^2 \cdot \{T - B(T) - \tfrac{1}{2}\kappa \cdot B(T)^2\}, \quad B(T) = (1 - e^{-\kappa T})/\kappa.
\end{aligned} \tag{4.3}$$

Proof. For simplicity in notation omit indices and set $\lambda = \lambda_{m+1}$, $W_t = W_t^{(m+1)}$. Integrating both sides of (4.2) over the interval $[0, T]$ yields the identity

$$r(T) - r(0) = \kappa\left(\theta + \lambda\sigma\kappa^{-1}\right) \cdot T - \kappa \cdot \int_0^T r(s)ds + (\sigma / \kappa) \cdot \int_0^T dW_s \ .$$

Identifying the first integral with the integrated process one gets the stochastic representation

$$R_T = \left(\theta + \lambda\sigma\kappa^{-1}\right) \cdot T + \kappa^{-1} \cdot \{r(0) - r(T) + \sigma \cdot W_T\}.$$

On the other hand, solving the stochastic differential equation (4.2), shows that

$$r(T) = r(0) \cdot e^{-\kappa T} + \left(\theta + \lambda\sigma\kappa^{-1}\right) \cdot (1 - e^{-\kappa T}) + \sigma \cdot \int_0^T e^{-\kappa(T-s)} dW_s \ .$$

Inserted into the preceding relation yields the (short rate independent) representation

$$R_T = \left(\theta + \lambda\sigma\kappa^{-1}\right) \cdot T + (r(0) - \theta - \lambda\sigma\kappa^{-1}) \cdot B(T) + (\sigma / \kappa) \cdot \left\{ W_T - \int_0^T e^{-\kappa(T-s)} dW_s \right\}.$$

Formulas for the first terms of the identities F1 and F2 follow at once, namely

$$\begin{aligned} &E[R_T] = \left(\theta + \lambda\sigma\kappa^{-1}\right) \cdot T + (r(0) - \theta - \lambda\sigma\kappa^{-1}) \cdot B(T), \\ &E^Q[R_T] = \theta \cdot T + (r(0) - \theta) \cdot B(T). \end{aligned}$$

Clearly, the risk-neutral integrated mean in F2 follows by setting $\lambda = 0$ in the real-world integrated mean in F1. To calculate the variance of the integrated process one notes that

$$\begin{aligned}\mathrm{Var}[R_T] &= (\sigma/\kappa)^2 \cdot \mathrm{Var}[W_T - \int_0^T e^{-\kappa(T-s)} dW_s] \\ &= (\sigma/\kappa)^2 \cdot \left\{ T - 2e^{-\kappa T} \cdot Cov[W_T, \int_0^T e^{\kappa s} dW_s] + e^{-2\kappa T} \cdot Var[\int_0^T e^{\kappa s} dW_s] \right\}.\end{aligned}$$

But, one has

$$\begin{aligned}Var[\int_0^T e^{\kappa s} dW_s] &= \int_0^{\mathrm{T}} e^{2\kappa s} ds = (1/2\kappa) \cdot (e^{2\kappa T} - 1), \\ Cov[W_T, \int_0^T e^{\kappa s} dW_s] &= \int_0^{\mathrm{T}} e^{\kappa s} ds = (1/\kappa) \cdot (e^{\kappa T} - 1)\end{aligned}.$$

Insert into the preceding relation and rearrange to get formula F3. The remaining covariance is calculated as follows (use the explicit representation of R_T):

$$Cov[R_T, W_T] = \mathrm{E}[R_T W_T] = (\sigma/\kappa) \cdot \mathrm{E}[W_T^2 - e^{-\kappa T} W_T \cdot \int_0^T e^{\kappa s} dW_s] = (\sigma/\kappa) \cdot \{T - B(T)\}.$$

Now, insert into the right-hand side of F1 and compare with the above integrated mean formula to show the identity F1. ◊

We are ready for the main result.

Theorem 4.1. (European option for the multidimensional Black-Scholes market with Vasicek interest rates) *Under the assumptions of Theorem* 2.1, *the market value at initial time of a European call option on the risky asset with real-world price* $S_t^{(k)}, k \in \{1,...,m\}$, *strike time* T *and strike price* K *is given by the closed-form formula*

$$E[D_T^{(m+1)}(S_T^{(k)} - K)_+] = S_0^{(k)} \cdot \Phi(d_1^{(k)}) - K \cdot \exp\{-r_f(T)\} \cdot \Phi(d_2^{(k)}), \qquad (4.4)$$

with

$$\begin{aligned}& d_1^{(k)} = \frac{\ln\{S_0^{(k)}/K\} + r_f(T) + \frac{1}{2}\upsilon_k^2(T)}{\upsilon_k(T)}, \quad d_2^{(k)} = d_1^{(k)} - \upsilon_k(T), \\ & r_f(T) = E^Q[R_T] - \tfrac{1}{2}\mathrm{Var}[R_T], \qquad (4.5) \\ & \upsilon_k^2(T) = \sigma_k^2 \cdot T + Var[R_T] + 2\rho_{1,m+1}\sigma_{\mathrm{k}} \cdot Cov[R_T, W_T^{(m+1)}],\end{aligned}$$

and these quantities are determined by Lemma 4.1.

Proof. By symmetry it suffices to show (4.4) for $k=1$. The random variables

$$X_1=\ln\left\{D_T^{(m+1)}S_T^{(1)}\right\}=\ln S_0^{(1)}-\tfrac{1}{2}\beta^T C\beta\cdot T-\beta^T W_T+(\lambda_1\sigma_1-\tfrac{1}{2}\sigma_1^2)\cdot T+\sigma_1\cdot W_T^{(1)}, \text{ and}$$

$$X_2=\ln\left\{D_T^{(m+1)}K\right\}=\ln K-R_T-\tfrac{1}{2}\beta^T C\beta\cdot T-\beta^T W_T$$

are normally distributed. Let $(\mu_1,\upsilon_1,\mu_2,\upsilon_2,\rho)$ denote their means, standard deviations and correlation coefficient. From the proof of the Theorem 2.1 one has

$$\mu_1=\ln S_0^{(1)}-\tfrac{1}{2}\beta^T C\beta\cdot T+(\lambda_1\sigma_1-\tfrac{1}{2}\sigma_1^2)\cdot T,\quad \upsilon_1^2=\beta^T C\beta\cdot T+(\sigma_1^2-2\lambda_1\sigma_1)\cdot T, \text{ hence}$$

$$\mu_1+\tfrac{1}{2}\upsilon_1^2=\ln S_0^{(1)}.$$

Through further calculation one obtains (use the identity F1 of Lemma 4.1)

$$\begin{aligned}
\mu_2&=\ln K-E[R_T]-\tfrac{1}{2}\beta^T C\beta\cdot T\\
&=\ln K-E^Q[R_T]-\lambda_{m+1}\cdot Cov[R_T,W_T^{(m+1)}]-\tfrac{1}{2}\beta^T C\beta\cdot T,\\
\upsilon_2^2&=Var[R_T]+\beta^T C\beta\cdot T+2\cdot\sum_j\rho_{j,m+1}\beta_j\cdot Cov[R_T,W_T^{(m+1)}]\\
&=Var[R_T]+\beta^T C\beta\cdot T+2\lambda_{m+1}\cdot Cov[R_T,W_T^{(m+1)}],
\end{aligned}$$

which implies that

$$\mu_2+\tfrac{1}{2}\upsilon_2^2=\ln K-E^Q[R_T]+\tfrac{1}{2}Var[R_T].$$

With

$$X_1-X_2=\ln\left\{S_0^{(1)}/K\right\}+(\lambda_1\sigma_1-\tfrac{1}{2}\sigma_1^2)\cdot T+\sigma_1\cdot W_T^{(1)}+R_T,$$

one deduces that

$$\begin{aligned}
\upsilon_1^2+\upsilon_2^2-2\rho\upsilon_1\upsilon_2&=Var[X_1-X_2]=Var[\sigma_1\cdot W_T^{(1)}+R_T]\\
&=\sigma_1^2\cdot T+Var[R_T]+2\rho_{1,m+1}\sigma_1\cdot Cov[R_T,W_T^{(m+1)}]=\upsilon_1^2(T).
\end{aligned}$$

Similarly, one gets

$$\begin{aligned}
&\upsilon_1^2 - \rho\upsilon_1\upsilon_2 = Cov[X_1, X_1 - X_2] = Cov[\sigma_1 \cdot W_T^{(1)} - \beta^T W_T, \sigma_1 \cdot W_T^{(1)} + R_T] \\
&= (\sigma_1^2 - \sum_j \rho_{1j}\beta_j\sigma_1)\cdot T + (\rho_{1,m+1}\sigma_1 - \sum_j \rho_{j,m+1}\beta_j)\cdot Cov[R_T, W_T^{(m+1)}] \\
&= (\sigma_1^2 - \lambda_1\sigma_1)\cdot T + (\rho_{1,m+1}\sigma_1 - \lambda_{m+1})\cdot Cov[R_T, W_T^{(m+1)}],
\end{aligned}$$

from which one gets

$$\begin{aligned}
&\mu_1 - \mu_2 + \upsilon_1^2 - \rho\upsilon_1\upsilon_2 \\
&= \ln\left\{S_0^{(1)} / K\right\} + (\lambda_1\sigma_1 - \tfrac{1}{2}\sigma_1^2)\cdot T + E^Q[R_T] + \lambda_{m+1}\cdot Cov[R_T, W_T^{(m+1)}] \\
&+ (\sigma_1^2 - \lambda_1\sigma_1)\cdot T + (\rho_{1,m+1}\sigma_1 - \lambda_{m+1})\cdot Cov[R_T, W_T^{(m+1)}] \\
&= \ln\left\{S_0^{(1)} / K\right\} + E^Q[R_T] - \tfrac{1}{2}\mathrm{Var}[R_T] \\
&+ \tfrac{1}{2}(\sigma_1^2 \cdot T + \mathrm{Var}[R_T] + 2\rho_{1,m+1}\sigma_1 \cdot Cov[R_T, W_T^{(m+1)}]) \\
&= \ln\left\{S_0^{(1)} / K\right\} + r_f(T) + \tfrac{1}{2}\upsilon_1^2(T),
\end{aligned}$$

and finally

$$\begin{aligned}
&\mu_1 - \mu_2 - \upsilon_2^2 + \rho\upsilon_1\upsilon_2 = (\mu_1 - \mu_2 + \upsilon_1^2 - \rho\upsilon_1\upsilon_2) - (\upsilon_1^2 + \upsilon_2^2 - 2\rho\upsilon_1\upsilon_2) \\
&= \ln\left\{S_0^{(k)} / K\right\} + r_f(T) - \tfrac{1}{2}\upsilon_1^2(T).
\end{aligned}$$

Inserting the obtained expressions into (3.2) one obtains (4.4) for $k = 1$. ◊

At this point some comments are in order. First of all, one notes that the defined Vasicek accumulated interest rate in (4.5) is closely related to bond pricing under the Vasicek affine model of the term structure of interest rates. Indeed, given the well-known bond pricing formula under the Q-measure, one verifies easily the relationship

$$\begin{aligned}
&B_0^T = \exp\{-A(T) - B(T)\cdot r(0)\} = \exp\{-r_f(T)\}, \\
&A(T) = \theta \cdot \{T - B(T)\} - \tfrac{1}{2}\mathrm{Var}[R_T], \quad B(T) = (1 - e^{-\kappa T})/\kappa.
\end{aligned} \tag{4.6}$$

It is interesting to compare (4.4) with the formula stated in Kim (2002). For this, start alternatively from (4.1) as real-world P- measure to get the following short rate dynamics under the Q-measure (e.g. Munk (2011), Section 7.4, equation (7.35))

$$dr(t) = \kappa\left(\theta - \lambda_{m+1}\sigma\kappa^{-1} - r(t)\right)\cdot dt + \sigma \cdot dW_t^Q . \tag{4.7}$$

Kim derives essentially the same closed-form formula through application of the equivalent martingale measure. Indeed, replacing $\theta - \lambda_{m+1}\sigma\kappa^{-1}$ by θ (to be consistent with the P-measure) and doing the necessary algebraic manipulations shows equality of the two formulas. This earlier derivation applies equally simple algebra and the Kunitomo and Takahashi (1992) Lemma, which is closely related to the bivariate version of Lemma 2.3 in [26]. Besides its attractive elementary approach, the state-price deflator method yields new insight, namely the validity of the formula under the presence of multiple Black-Scholes risky assets that are all correlated with Vasicek interest rates, and the invariance of it against changing market prices of risk. From an economic point of view, the latter means that, in contrast to the CAPM by Treynor (1961/62), Sharpe (1964), Lintner (1965) and Mossin (1966) for investment in stocks portfolios only, option pricing does not reward for taking risks. Actually, similar formulas could also be derived this way for any other Gaussian short rate processes including the Merton-Ho-Lee type (continuous-time version of the Ho and Lee (1986) model with constant drift). Closed-form formulas for Gaussian processes have originally been suggested by Merton (1973), Rabinovitch (1989), and Amin and Jarrow (1992). The state-price deflator approach has been pioneered by Rubinstein (1976) who recovered this way the Black-Scholes formula (consult Yao (2001) for a wealth of references about this method). By passing, we note that Kim (2002) proposes some interesting closed-form approximation formulas for non-Gaussian short rate processes as the Cox-Ingersoll-Ross (1985) affine model and the Brennan and Schwartz (1980) non-affine model, the latter being also considered in Courtadon (1982). These asymptotic approximations are based on the approach by Kim and

Kunitomo (1999), which has been validated in Kunitomo and Takahashi (2003) (see also Takahashi (2009)). Fang [17],[18] uses the partial differential equation approach to derive a pricing formula under Vasicek interest rates. However, the latter technique applies much more involved mathematics. Other papers include Wilhelm (2001), and Abudy and Izhakian (2011). The state-price deflator approach adds another technique to the eight valuation methods identified by Andreasen et al. (1998) that can be used to derive the original Black-Scholes formula. Note that further generalizations are possible. Among the hybrid models with risky assets and both stochastic volatility and interest rates, one might mention Guohe (2007), who derives closed-form pricing formulas for a double exponential jump-diffusion model with Vasicek and CIR interest rates, and Grzelak and Oosterlee (2011) who obtain some general approximations. Option pricing in an international economy with additional stochastic FX rates is of further interest (e.g. Haastrecht and Pelsser (2011), Wittke (2011)). Finally, research on further alternative representations of the Black-Scholes formula is of equally current interest. For example, Madan et al. (2008) express it as cumulative function of a last (or first) passage time of Brownian motion (see also Profeta et al. (2010)). Without doubt, investigations related to the important Black-Scholes milestone will continue over the years to enrich both Mathematics and Finance.

REFERENCES

[1] M. Abudy, Y. Izhakian, Pricing stock options with stochastic interest rate, preprint (2011), URL: http://ssrn.com/abstract=1944450

[2] K.I. Amin, R.A. Jarrow, Pricing options on risky assets in a stochastic interest rate economy, Mathematical Finance 4, (1992), 217-237.

[3] J. Andreasen, B. Jensen, R. Poulsen (1998). Eight valuation methods in financial mathematics: the Black-Scholes formula as an example, Math. Scientist 23, (1998), 18-40.

[4] K. Arrow, An extension of the basic theorems of classical welfare economics. In: Proc. 2nd Berkeley Symp. Math. Statist. Probab., 507-532, University of California Press, 1951.

[5] K. Arrow, Le role des valeurs boursières pour la répartition la meilleure des risques, Econometrica 40, (1953), 41-47 (English translation [6]).

[6] K. Arrow, The role of securities in the optimal allocation of risk-bearing, Review of Economic Studies 31, (1964), 91-96.

[7] K. Arrow, Essays in the Theory of Risk-Bearing, North-Holland, 1971.

[8] M. Bhattacharya, G.M. Constantinides, Theory of Valuation: Frontier of Modern Financial Theory, Rowman & Littlefield, Totowa, NJ, 1989.

[9] C. Bernard, Z. Cui, A note on exchange options under stochastic interest rates, preprint (2010), URL: http://ssrn.com/abstract=1626020.

[10] F. Black, M. Scholes (1973). The pricing of options and corporate liabilities, J. of Political Economy 81, (1973), 637-59, Reprinted in [23].

[11] M.J. Brennan, E.S. Schwartz, Analyzing convertible bonds, J. of Financial and Quantitative Analysis 15, (1980), 907-929.

[12] J. Cox, J. Ingersoll, S. Ross, A theory of the term structure of interest rates, Econometrica 53, (1985), 385-408.

[13] G. Courtadon, The pricing of options on default-free bonds, J. of Financial and Quantitative Analysis 17(1), (1982), 75-100.

[14] G. Debreu, Valuation equilibrium and Pareto optimum, Proceedings of the National Academy of Sciences 40, (1954), 588-592.

[15] E. Dimson, M. Mussavian, Three centuries of asset pricing. J. of Banking and Finance 23, (1999) 1745-1769.

[16] D. Duffie, Dynamic Asset Pricing Theory, Princeton University Press, NJ, 1992.

[17] H. Fang, Option pricing under stochastic interest rates, Int. J. of Engineering and Manufacturing 3, (2012), 82-89.

[18] H. Fang, European option pricing formula under stochastic interest rate, Progress in Applied Mathematics 4(1), (2012), 14-21.

[19] L.A. Grzelak, C.W. Oosterlee, On the Heston model with stochastic interest rates, SIAM J. Financial Math. 2, (2011), 255-286.

[20] D. Guohe, Pricing European option in a double exponential-jump diffusion model with two market structure risks, Appl Math. J. Chinese Univ. Ser. B 22(2), (2007), 127-137.

[21] A. van Haastrecht, A. Pelsser, Generic pricing of FX, inflation and stock options under stochastic interest rates and stochastic volatility, Quant. Finance 11(5), (2011), 665-691.

[22] T.S. Ho, S.-B. Lee, Term structure movements and pricing interest rate contingent Claims, The J. of Finance 41, (1986), 1011-1029.

[23] L. Hugston (Editor), Options: Classic Approaches to Pricing and Modelling, Risk Books, 1999.

[24] W. Hürlimann, An extension of the Black-Scholes and Margrabe formulas to a multiple risk economy, Applied Mathematics 2(4), (2011), 137-142.

[25] W. Hürlimann, Analytical pricing of an insurance embedded option: alternative formulas and Gaussian approximation, J. Informatics and Math. Sci. 3(2), (2011), 87-105.

[26] W. Hürlimann, The algebra of option pricing: theory and application, Int. J. of Algebra and Statistics 1(2), (2012), 68-88.

[27] M. Jeanblanc, M. Yor, M. Chesney, Mathematical Methods for Financial Markets, Springer Finance, Springer-Verlag London Limited, 2009.

[28] Y.-J. Kim, Option pricing under stochastic interest rates: an empirical investigation, Asia-Pacific Financial Markets 9(1), (2002), 23-44.

[29] Y.-J. Kim, N. Kunimoto, Pricing options under stochastic interest rates: a new Approach, Asia-Pacific Financial Markets 6(1), (1999), 49-70.

[30] N. Kunitomo, A. Takahashi, Pricing average options (in Japanese), Japan Financial Review 14, (1992), 1-19.

[31] N. Kunitomo, A. Takahashi, On validity of the asymptotic expansion approach in contingent claim analysis, Annals of Applied Probability 13(3), (2003), 914-952.

[32] J. Lintner, The valuation of risky assets and the selection of risky investments in stocks portfolios and capital budgets, Review of Economics and Statistics 47(1), (1965), 13-37.

[33] D. Madan, B. Roynette, M. Yor, An alternative expression for the Black-Scholes formula in terms of Brownian first and last passage times, preprint (2008),
URL: http://hal.archives-ouvertes.fr/hal-00257403/en/

[34] W. Margrabe, The value of an option to exchange one asset for another., J. of Finance 33, (1978), 177-186, Reprinted in Hugston [23].

[35] R.C. Merton, Theory of Rational Option Pricing, Bell J. of Economics and Management Science 4(1), (1973), 141–183. Reprinted in [36].

[36] R.C. Merton, Continuous-Time Finance, Basil Blackwell, 1990.

[37] K.R. Milterssen, S.A. Persson, Pricing rate of return guarantees in a Heath-Jarrow-Morton

framework, Insurance: Mathematics and Economcis 25(3), (1999), 307-325.

[38] J. Mossin, Equilibrium in a capital asset market, Econometrica 34(4), (1966), 768-783.

[39] C. Munk, Fixed Income Modelling, Oxford University Press, New York, 2011.

[40] T. Negishi, Welfare economics and existence of an equilibrium for a competitive Economy, Metroeconometrica 12, (1960), 92-97.

[41] R. Poulsen, The Margrabe formula, Encyclopedia of Quantitative Finance. J. Wiley, 2010.

[42] C. Profeta, B. Roynette, M. Yor, Option prices as probabilities: a new look at generalized Black-Scholes formulae, Springer Finance, Berlin & Heidelberg, 2010.

[43] R. Rabinovitch, Pricing stock and bond options when the default-free rate is stochastic, J. of Financial and Quantitative Analysis 24, (1989), 447-457.

[44] S.A. Ross, A simple approach to the valuation of risky streams, J. of Business 51, (1978), 453-475.

[45] M. Rubinstein, The valuation of uncertain income streams and the pricing of options. Bell J. of Economics 7, (1976), 407-425. Reprinted in [8].

[46] W.F. Sharpe, Capital asset prices: a theory of market equilibrium under conditions of Risk, J. of Finance 19(3), (1964), 425-442.

[47] N. Takahashi, On an asymptotic expansion approach to numerical problems in Finance, Selected Papers on Probability and Statistics, 199-217. American Mathematical Society, 2009.

[48] J.L. Treynor, Unpublished manuscript (1961).

[49] J.L. Treynor, Toward a theory of market value of risky assets, manuscript (1962), In: R.A. Korajczyk (Ed.), Asset Pricing and Portfolio Performance: Models, Strategy and Performance Metrics, 15-22, Risk Books, London, 1999.

[50] O.A. Vasicek, An equilibrium characterization of the term structure, J. of Financial Economics 5, (1977), 177-188.

[51] J. Wilhelm, Option prices with stochastic interest rates – Black-Scholes and Ho-Lee Unified, Discussion paper no. B-8-01. (2001), URL: http://www.wiwi.uni-passau.de/1125.html

[52] M. Wittke, Essays about option valuation under stochastic interest rates, Ph.D. Thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, 2011. URL: http://hss.ulb.uni-bonn.de/2011/2485/2485.htm

[53] Y. Yao, State price density, Esscher transforms, and pricing options on stocks, bonds and foreign exchange rates, North American Actuarial Journal 5(3), (2001), 104-117.