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GENERALIZED OPEN SETS IN HEREDITARY GENERALIZED TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and study the notions of $B_{\mathscr{H}}$-sets, $C_{\mathscr{H}}$-sets and $S_{\mathscr{H}}$-sets and also we study the properties of generalized open sets in hereditary generalized topological spaces.

Keywords: Hereditary generalized topology; α-$\mathscr{H}$-open; β-$\mathscr{H}$-open; π-$\mathscr{H}$-open.

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1. Introduction and preliminaries

In 2002, Csaszar [1] introduced the notions of generalized topology and generalized continuity. A generalized topology (breifly GT) [1] μ on a non-empty set X is a collection of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary union. Here, a generalized topology μ is said to be strong [2], if $X \in \mu$. (X,μ) is called a quasi topology [3] if μ is closed under finite intersection. A subset A of a generalized topological space (X,μ) is called μ-σ-open [4] (resp. μ-π-open [4], μ-α-open [4], μ-β-open [4], μ-δ-open [5], μ-b-open [6], μ-t^*-open [7]) if $A \subset c_\mu(i_\mu(A))$ (resp. $A \subset i_\mu(c_\mu(A))$ $A \subset i_\mu(c_\mu(i_\mu(A)))$, $A \subset c_\mu(i_\mu(c_\mu(A)))$, $i_\mu(c_\mu(A)) \subset c_\mu(i_\mu(A))$, $A \subset i_\mu(c_\mu(A)) \cup c_\mu(i_\mu(A))$, $i_\mu(c_\mu(i_\mu(A))) = i_\mu(A)$).

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A hereditary class $\mathscr{H}$ of X is a non-empty collection of subsets of X such that $A \subset B$, $B \in \mathscr{H}$ implies $A \in \mathscr{H}$ [5]. With respect to the generalized topology μ of all μ-open sets and a hereditary class $\mathscr{H}$, for each subset A of X, a subset $A^*(\mathscr{H})$ or simply A^* of X is defined by A^*=$\{x \in X / M \cap A \notin \mathscr{H}$ for every $M \in \mu$ containing $x\}$ [5]. $\mathscr{H}$ is said to be μ-codense[8], if $\mu \cap \mathscr{H} = \{\emptyset\}$ and is said to be completely μ-codense [8], if $\pi(X) \cap \mathscr{H} = \{\emptyset\}$, where $\pi(X)$ is the family of all π-open sets in (X, μ). Then $c^*_\mu(A) = A \cup A^*$ for every subset A of X, with respect to μ and hereditary $\mathscr{H}$ class of subsets of X.

Lemma 1.1. [8] *If (X, μ) is a quasi topology with a hereditary class $\mathscr{H}$. Then the following are hold.*

(a) $\mathscr{H}$ is μ-codense if and only if $A \subset A^*$ for every $A \in \mu$.

(b) If $A \subset A^*$, then $A^* = c_\mu(A^*) = c_\mu(A) = c^*_\mu(A)$.

Lemma 1.2. [8] *Let (X, μ) be a generalized topological space with a hereditary class $\mathscr{H}$. Then $\mathscr{H}$ is completely μ-codense if and only if $\mathscr{H} \subset \mathscr{N}$, where $\mathscr{N}$ is the hereditary class of all nowhere dense sets in (X, μ).*

2. Generalized open sets in HGTS

Definition 2.1. A subset A of a hereditary generalized topological space $(X, \mu, \mathscr{H})$ is said to be

a) α-$\mathscr{H}$-open [5] if $A \subseteq i_\mu(c^*_\mu(i_\mu(A)))$,

b) β-$\mathscr{H}$-open [5] if $A \subseteq c_\mu(i_\mu(c^*_\mu(A)))$,

c) σ-$\mathscr{H}$-open [5] if $A \subseteq c^*_\mu(i_\mu(A))$,

d) π-$\mathscr{H}$-open [5] if $A \subseteq i_\mu(c^*_\mu(A))$,

e) δ-$\mathscr{H}$-open [8] if $i_\mu(c^*_\mu(A)) \subset c^*_\mu(i_\mu(A))$,

f) strong β-$\mathscr{H}$-open [5] if $A \subseteq c^*_\mu(i_\mu(c^*_\mu(A)))$,

g) μ^*-closed [5] if $c^*_\mu(A) \subset A$.

Definition 2.2. A subset A of a hereditary generalized topological space $(X, \mu, \mathscr{H})$ is said to be b-$\mathscr{H}$-open if $A \subseteq i_\mu(c^*_\mu(A)) \cup c^*_\mu(i_\mu(A))$.

Proposition 2.3. *Let $(X, \mu, \mathscr{H})$ be a hereditary generalized topological space and $A \subset X$. Let $A = B \cup C$, where B is π-$\mathscr{H}$-open and C is σ-$\mathscr{H}$-open. Then A is b-$\mathscr{H}$-open.*

Proof. The proof is obvious.

Proposition 2.4. *In a hereditary generalized topological space $(X,\mu,\mathscr{H})$ the following holds:*

a) Every α-$\mathscr{H}$-open set is μ-α-open.

b) Every σ-$\mathscr{H}$-open set is μ-σ-open.

c) Every β-$\mathscr{H}$-open set is μ-β-open.

d) Every b-$\mathscr{H}$-open set is μ-b-open.

Proof. *a)* Let A be an α-$\mathscr{H}$-open set. Then $A \subset i_\mu(c^*_\mu(i_\mu(A))) = i_\mu((i_\mu(A))^* \cup i_\mu(A)) \subset i_\mu((c_\mu(i_\mu(A)) \cup i_\mu(A)) = i_\mu(c_\mu(i_\mu(A)))$. Hence A is μ-α-open.

b) Let A be a σ-$\mathscr{H}$-open set. Then $A \subset c^*_\mu(i_\mu(A)) = (i_\mu(A))^* \cup i_\mu(A) \subset c_\mu(i_\mu(A))$. Hence A is μ-σ-open.

c) Let A be a β-$\mathscr{H}$-open set. Then $A \subset c_\mu(i_\mu(c^*_\mu(A))) = c_\mu(i_\mu(A^* \cup A)) \subset c_\mu(i_\mu(c_\mu(A) \cup A)) = c_\mu(i_\mu(c_\mu(A)))$. Hence A is μ-β-open.

d) Let A be b-$\mathscr{H}$-open set. Then $A \subset i_\mu(c^*_\mu(A)) \cup c^*_\mu(i_\mu(A)) = i_\mu(A^* \cup A) \cup ((i_\mu(A))^* \cup i_\mu(A))$ $\subset i_\mu(c_\mu(A) \cup (A)) \cup (c_\mu(i_\mu(A) \cup i_\mu(A))) = i_\mu(c_\mu(A)) \cup c_\mu(i_\mu(A))$. Hence A is μ-b-open.

Remark 2.5. The following examples show that the converse of Proposition 2.4, need not be true.

Example 2.6. *Let $X = \{a,b,c,d,e\}$, $\mu = \{\emptyset,\{a\},\{c\},\{a,c\},\{a,b,c\},\{c,d,e\},\{a,c,d,e\},X\}$ and $\mathscr{H} = \{\emptyset,\{a\},\{b\},\{c\}\}$. Then $A = \{c,e\}$ is μ-α-open (resp. μ-σ-open) but not α-$\mathscr{H}$-open (resp. σ-$\mathscr{H}$-open).*

Example 2.7. *Let $X = \{a,b,c,d\}$, $\mu = \{\emptyset,\{a,b\},\{a,c\},\{b,c\},\{c,d\},\{a,b,c\},\{a,c,d\},$ $\{b,c,d\},X\}$ and $\mathscr{H} = \{\emptyset,\{c\},\{d\}\}$. Then $A = \{c\}$ is μ-β-open but not β-$\mathscr{H}$-open.*

Example 2.8. *Let $X = \{a,b,c,d,e\}$, $\mu = \{\emptyset,\{a\},\{c\},\{a,c\},\{a,b,c\},\{c,d,e\},\{a,c,d,e\},X\}$ and $\mathscr{H} = \{\emptyset,\{a\},\{b\},\{c\}\}$. Then $A = \{a,b\}$ is μ-b-open but not b-$\mathscr{H}$-open.*

Proposition 2.9. *In a hereditary generalized topological space $(X,\mu,\mathscr{H})$ the following holds:*

a) Every α-$\mathscr{H}$-open set is π-$\mathscr{H}$-open.

b) Every α-$\mathscr{H}$-open set is σ-$\mathscr{H}$-open.

c) Every π-$\mathscr{H}$-open set is β-$\mathscr{H}$-open.

d) Every σ-$\mathscr{H}$-open set is b-$\mathscr{H}$-open.

e) Every π-$\mathscr{H}$-open set is b-$\mathscr{H}$-open.

f) Every b-$\mathscr{H}$-open set is β-$\mathscr{H}$-open.

Proof. *a)* Let A be α-$\mathscr{H}$-open. Then $A \subset i_\mu(c^*_\mu(i_\mu(A))) \subset i_\mu(c^*_\mu(A))$. Hence A is π-$\mathscr{H}$-open.

b) Let A be α-$\mathscr{H}$-open. Then $A \subset i_\mu(c^*_\mu(i_\mu(A))) \subset c^*_\mu(i_\mu(A))$. Hence A is σ-$\mathscr{H}$-open.

c) Let A be π-$\mathscr{H}$-open. Then $A \subset i_\mu(c^*_\mu(A)) \subset c_\mu(i_\mu(c^*_\mu(A)))$. Hence A is β-$\mathscr{H}$-open.

d) Let A be σ-$\mathscr{H}$-open. Then $A \subset c^*_\mu(i_\mu(A)) \subset i_\mu(c^*_\mu(A)) \cup c^*_\mu(i_\mu(A))$. Hence A is b-$\mathscr{H}$-open.

e) Let A be π-$\mathscr{H}$-open. Then $A \subset i_\mu(c^*_\mu(A)) \subset i_\mu(c^*_\mu(A)) \cup c^*_\mu(i_\mu(A))$. Hence A is b-$\mathscr{H}$-open.

f) Let A be b-$\mathscr{H}$-open. Then $A \subset i_\mu(c^*_\mu(A)) \cup c^*_\mu(i_\mu(A)) \subset c_\mu(i_\mu(c^*_\mu(A)) \cup c^*_\mu(i_\mu(A))) = c_\mu(i_\mu(c^*_\mu(A))) \cup c_\mu(c^*_\mu(i_\mu(A))) = c_\mu(i_\mu(c^*_\mu(A))) \cup c_\mu((i_\mu(A))^* \cup i_\mu(A))$
$\subset c_\mu(i_\mu(c^*_\mu(A))) \cup c_\mu(c_\mu(i_\mu(A)) \cup i_\mu(A)) = c_\mu(i_\mu(c^*_\mu(A))) \cup c_\mu(i_\mu(A)) \subset c_\mu(i_\mu(c^*_\mu(A)))$. Hence A is β-$\mathscr{H}$-open.

Remark 2.10. The following examples show that the converse of Proposition 2.9, need not be true.

Example 2.11. *Let $X = \{a,b,c,d\}$, $\mu = \{\emptyset, \{a,c\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \{a,c,d\}, X\}$ and $\mathscr{H} = \{\emptyset, \{a\}, \{b\}\}$. Then $A = \{c,d\}$ is π-$\mathscr{H}$-open but not α-$\mathscr{H}$-open.*

Example 2.12. *Let $X = \{a,b,c,d\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{b,c,d\}, X\}$ and $\mathscr{H} = \{\emptyset, \{a\}, \{b\}\}$. Then $A = \{a,b,d\}$ is σ-$\mathscr{H}$-open but not α-$\mathscr{H}$-open.*

Example 2.13. *Let $X = \{a,b,c,d,e\}$, $\mu = \{\emptyset, \{a\}, \{c\}, \{a,c\}, \{a,b,c\}, \{c,d,e\}, \{a,c,d,e\}, X\}$ and $\mathscr{H} = \{\emptyset, \{a\}, \{b\}, \{c\}\}$. Then $A = \{a,b\}$ is β-$\mathscr{H}$-open but not π-$\mathscr{H}$ open.*

Example 2.14. *Let $X = \{a,b,c,d\}$, $\mu = \{\emptyset, \{c\}, \{a,b\}, \{a,b,c\}, X\}$ and $\mathscr{H} = \{\emptyset, \{d\}\}$. Then $A = \{a,c\}$ is b-$\mathscr{H}$-open set but not σ- $\mathscr{H}$-open.*

Example 2.15. *Let $X = \{a,b,c,d\}$, $\mu = \{\emptyset, \{c\}, \{a,b\}, \{a,b,c\}, \{b,c,d\}, X\}$ and $\mathscr{H} = \{\emptyset, \{d\}\}$. Then $A = \{c,d\}$ is b-$\mathscr{H}$-open set but not π-$\mathscr{H}$-open.*

Example 2.16. *Let $X = \{a,b,c,d\}$, $\mu = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X\}$ and $\mathscr{H} = \{\emptyset, \{a\}, \{b\}\}$. Then $A = \{a,b,d\}$ is β-$\mathscr{H}$-open but not b-$\mathscr{H}$-open.*

Proposition 2.17. *Let $(X,\mu,\mathscr{H})$ be a hereditary generalized topological space. Let A and B be subsets of X. If A is an α-$\mathscr{H}$-open set and B is a β-$\mathscr{H}$-open set then $A\cap B$ is a β-$\mathscr{H}$-open set.*

Proof. Let A be an α-$\mathscr{H}$-open set and B be a β-$\mathscr{H}$-open set. Then $A\cap B \subset i_\mu(c^*_\mu(i_\mu(A)))\cap c_\mu(i_\mu(c^*_\mu(B))) \subset c_\mu(i_\mu(c^*_\mu(i_\mu(A)))\cap i_\mu(c^*_\mu(B))) \subset c_\mu(i_\mu(c^*_\mu(i_\mu(A))\cap i_\mu(c^*_\mu(B))))$
$= c_\mu(i_\mu(c^*_\mu(i_\mu(A)\cap B))) \subset c_\mu(i_\mu(c^*_\mu(A\cap B)))$. Hence $A\cap B$ is β-$\mathscr{H}$-open.

Proposition 2.18. *Let $(X,\mu,\mathscr{H})$ be a hereditary generalized topological space and $A\subset X$. Then A is σ-$\mathscr{H}$-open if and only if there exists a μ-open set U such that $U\subset A\subset c^*_\mu(U)$.*

Proof. Suppose that A is σ-$\mathscr{H}$-open. Then $A\subset c^*_\mu(i_\mu(A))$. Let $U=i_\mu(A)$. Then $U\subset A\subset c^*_\mu(U)$. Conversely, assume that there exists a μ-open set U such that $U\subset A\subset c^*_\mu(U)$. Then $U\subset i_\mu(A)$ and thus $c^*_\mu(U)\subset c^*_\mu(i_\mu(A))$, but $A\subset c^*_\mu(U)$, so $A\subset c^*_\mu(i_\mu(A))$, that is A is σ-$\mathscr{H}$-open.

Proposition 2.19. *Let $(X,\mu,\mathscr{H})$ be a hereditary generalized topological space and $A\subset X$. If there exists a π-$\mathscr{H}$-open set U such that $U\subset A\subset c^*_\mu(U)$, then A is strong β-$\mathscr{H}$-open.*

Proof. Since $U\subset A\subset c^*_\mu(U)$, it follow that $c^*_\mu(A)=c^*_\mu(U)$, and thus $i_\mu(c^*_\mu(A))=i_\mu(c^*_\mu(U))$. But U is π-$\mathscr{H}$-open, so $U\subset i_\mu(c^*_\mu(A))$ and $A\subset c^*_\mu(U)$, so $A\subset c^*_\mu(i_\mu(c^*_\mu(A)))$, that is, A is strong β-$\mathscr{H}$-open.

Remark 2.20. The following example shows that the converse of Proposition 2.19, need not be true.

Example 2.21. *Let $X=\{a,b,c,d,e,f,g\}$, $\mu=\{\emptyset,\{a,b,c\},\{c,d,e\},\{e,f,g\},\{a,b,c,d,e\},$ $\{a,b,c,e,f,g\},\{c,d,e,f,g\}\}$ and $\mathscr{H}=\{\emptyset,\{a\},\{b\}\}$. Then $A=\{a,b,d\}$ is strong β-$\mathscr{H}$-open.*
(i) If $U=\emptyset$, then U is π-$\mathscr{H}$-open, but $c^*_\mu(U)=\emptyset$.
(ii) If $U=\{a,b\}$, then U is not π-$\mathscr{H}$-open, because $i_\mu(c^*_\mu(U))=\emptyset$.
(iii) If $U=\{a,d\}$, then U is not π-$\mathscr{H}$-open, because $c^*_\mu(U)=\{a,d\}$ and $i_\mu(c^*_\mu(U))=\emptyset$.
(iv) If $U=\{b,d\}$, then U is not π-$\mathscr{H}$-open, because $c^*_\mu(U)=\{b,d\}$ and $i_\mu(c^*_\mu(U))=\emptyset$.
(v) If $U=\{a,b,d\}$, then U is π-$\mathscr{H}$-open, but $c^*_\mu(U)=\emptyset$.
Hence there is no π-$\mathscr{H}$-open set U such that $U\subset A\subset c^*_\mu(U)$.

Theorem 2.22. *Let $(X,\mu,\mathscr{H})$ be a hereditary generalized topological space and $A \subset X$. If A is both α-$\mathscr{H}$-open and μ^*-closed, then A is μ-open.*

Proof. Let A be α-$\mathscr{H}$-open set. Then $A \subset i_\mu(c^*_\mu(i_\mu(A))) \subset i_\mu(c^*_\mu(A)) \subset i_\mu(A)$. Hence A is μ-open.

Theorem 2.23. *Let (X,μ,H) be a hereditary generalized topological space and $A \subset X$. If A is both β-$\mathscr{H}$-open and μ^*-closed, then A is μ-σ-open.*

Proof. The proof is obvious.

Proposition 2.24. *Let $(X,\mu,\mathscr{H})$ be a hereditary generalized topological space and $A \subset X$. If A is both b-$\mathscr{H}$-open and μ^*-closed, then A is a σ-$\mathscr{H}$-open set.*

Proof. Let A be b-$\mathscr{H}$-open and μ^*-closed. Then $A \subset (i_\mu(c^*_\mu(A)) \cup c^*_\mu(i_\mu(A))) = (i_\mu(A) \cup ((i_\mu(A))^* \cup i_\mu(A))) = (i_\mu(A) \cup (i_\mu(A))^*) = c^*_\mu(i_\mu(A))$. Then A is a σ-$\mathscr{H}$-open set.

Proposition 2.25. *Let $(X,\mu,\mathscr{H})$ be a hereditary generalized topological space and $A \subset X$. If A is b-$\mathscr{H}$-open such that $i_\mu(A) = \{\emptyset\}$, then A is a π-$\mathscr{H}$-open.*

Proof. Let A be a b-$\mathscr{H}$-open set. Then $A \subset i_\mu(c^*_\mu(A)) \cup c^*_\mu(i_\mu(A)) = i_\mu(c^*_\mu(A))$. Hence A is a π-$\mathscr{H}$-open set.

Proposition 2.26. *Let $(X,\mu,\mathscr{H})$ be a hereditary generalized topological space and $A \subset X$. If A is a b-$\mathscr{H}$-open set, then A is a strong β-$\mathscr{H}$-open set.*

proof. *Let A be b-$\mathscr{H}$-open. Then $A \subset i_\mu(c^*_\mu(A)) \cup c^*_\mu(i_\mu(A)) \subset c^*_\mu(i_\mu(c^*_\mu(A))) \cup c^*_\mu(i_\mu(A)) = c^*_\mu(i_\mu(c^*_\mu(A)) \cup i_\mu(A)) \subset c^*_\mu(i_\mu(c^*_\mu(A) \cup A)) = c^*_\mu(i_\mu(c^*_\mu(A)))$. Hence A is a strong β-$\mathscr{H}$-open set.*

Remark 2.27. The following example shows that the converse of Proposition 2.26, need not be true.

Example 2.28. *Let $X = \{a,b,c,d\}$, $\mu = \{\emptyset,\{b\},\{a,d\},\{a,b,d\}\}$ and $\mathscr{H} = \{\emptyset,\{b\}\}$. Then $A = \{a,c\}$ is strong β-$\mathscr{H}$-open but not b-$\mathscr{H}$-open.*

Proposition 2.29. *Let $(X,\mu,\mathscr{H})$ be a hereditary generalized topological space and $A \subset X$. Then the following are equivalent.*

1. A is σ-$\mathscr{H}$-open.

2. A is both b-$\mathscr{H}$-open and δ-$\mathscr{H}$-open.

Proof. (1) $\Rightarrow$ (2). Let A be σ-$\mathscr{H}$-open. Then $A \subset c^*_\mu(i_\mu(A))$. Now $i_\mu(c^*_\mu(A)) \subset c^*_\mu(A) \subset c^*_\mu(c^*_\mu(i_\mu(A))) = c^*_\mu(i_\mu(A))$. Hence A is δ-$\mathscr{H}$-open set.
(2) $\Rightarrow$ (1). The proof is obvious.

Proposition 2.30. *Let $(X,\mu,\mathscr{H})$ be a hereditary generalized topological space and $A \subset X$. Let $\mathscr{H}$=P(X). Then A is b-$\mathscr{H}$-open if and only if A is μ-open.*

Proof. *If $\mathscr{H} = P(X)$, then $A^* = \emptyset$. If A is b-$\mathscr{H}$-open, then $A \subset i_\mu(c^*_\mu(A)) \cup c^*_\mu(i_\mu(A)) = i_\mu(A^* \cup A) \cup ((i_\mu(A))^* \cup i_\mu(A)) = i_\mu(A) \cup i_\mu(A) = i_\mu(A)$. Hence A is μ-open.*

Remark 2.31. The following examples show that the intersection of two *b*-$\mathscr{H}$-open sets (resp. α-$\mathscr{H}$-open sets, σ-$\mathscr{H}$-open sets, β-$\mathscr{H}$-open sets, π-$\mathscr{H}$-open sets) need not be *b*-$\mathscr{H}$-open set (resp. α-$\mathscr{H}$-open set, σ-$\mathscr{H}$-open set,β-$\mathscr{H}$-open set,π-$\mathscr{H}$-open set).

Example 2.32. *Let $X = \{a,b,c,d\}$, $\mu = \{\emptyset,\{a,b\},\{b,c\},\{a,b,c\},\{b,c,d\},X\}$ and $\mathscr{H} = \{\emptyset,\{a\},\{b\}\}$. Then $A = \{a,b\}$ and $B = \{b,c\}$ are b-$\mathscr{H}$-open sets. But $A \cap B = \{b\}$ is not b-$\mathscr{H}$-open set.*

Example 2.33. *Let $X = \{a,b,c,d\}$, μ=$\{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},\{a,b,c\}$, $\{b,c,d\},X\}$ and $\mathscr{H}$=$\{\emptyset,\{a\},\{b\}\}$. Then A=$\{a,c\}$ and B=$\{b,c\}$ are α-$\mathscr{H}$-open sets. But $A \cap B$=$\{c\}$ is not α-$\mathscr{H}$-open set (not σ-$\mathscr{H}$-open).*

Example 2.34. *Let $X = \{a,b,c,d\}$, μ=$\{\emptyset,\{c\},\{a,b\},\{b,c,d\},\{a,b,c\},X\}$ and $\mathscr{H}$=$\{\emptyset,\{d\}\}$. Then*

1. A=$\{a,b,d\}$ and B=$\{a,c,d\}$ are β-$\mathscr{H}$-open sets. But $A \cap B$=$\{a,d\}$ is not β-$\mathscr{H}$-open.

2. A=$\{a,b\}$ and B=$\{a,c\}$ are π-$\mathscr{H}$-open sets. But $A \cap B$=$\{a\}$ is not π-$\mathscr{H}$-open.

Proposition 2.35. *Let $(X,\mu,\mathscr{H})$ be a hereditary generalized topological space. Let A and B be subsets of X. If A is an α-$\mathscr{H}$-open set and B is a b-$\mathscr{H}$-open set, then $A \cap B$ is a b-$\mathscr{H}$-open set.*

Proof. By hypothesis, $A \subset i_\mu(c^*_\mu(i_\mu(A)))$ and $B \subset i_\mu(c^*_\mu(B)) \cup c^*_\mu(i_\mu(B))$.
Now $A \cap B \subset i_\mu(c^*_\mu(i_\mu(A))) \cap (i_\mu(c^*_\mu(B)) \cup c^*_\mu(i_\mu(B)))$
$=(i_\mu(c^*_\mu(i_\mu(A))) \cap i_\mu(c^*_\mu(B))) \cup (i_\mu(c^*_\mu(i_\mu(A))) \cap c^*_\mu(i_\mu(B)))$

$\subset i_\mu(c^*_\mu(i_\mu(A)) \cap i_\mu(c^*_\mu(B))) \cup c^*_\mu(i_\mu(c^*_\mu(i_\mu(A))) \cap (i_\mu(B)))$

$\subset i_\mu(c^*_\mu(i_\mu(A) \cap i_\mu(c^*_\mu(B)))) \cup c^*_\mu(i_\mu(c^*_\mu(i_\mu(A)) \cap (i_\mu(B))))$

$\subset i_\mu(c^*_\mu(i_\mu(i_\mu(A) \cap c^*_\mu(B)))) \cup c^*_\mu(i_\mu(c^*_\mu(i_\mu(A)) \cap (i_\mu(B))))$

$\subset i_\mu(c^*_\mu(i_\mu(c^*_\mu(A \cap B)))) \cup c^*_\mu(i_\mu(c^*_\mu(i_\mu(A \cap B))))$

$\subset i_\mu(c^*_\mu(A \cap B)) \cup c^*_\mu(i_\mu(A \cap B))$. Hence $A \cap B$ is a b-$\mathscr{H}$-open set.

Theorem 2.36. *Let (X, μ) be a quasi topology with a hereditary class $\mathscr{H}$ and $\mathscr{H}$ be μ-completely codense. Then A is b-$\mathscr{H}$-open if and only if A is μ-b-open.*

Proof. Clearly, every b-$\mathscr{H}$-open set is a μ-b-open set. Conversely, suppose A is μ-b-open. Then $A \subset i_\mu(c_\mu(A)) \cup c_\mu(i_\mu(A))$. Since $\mathscr{H}$ is completely μ-codense, $\mathscr{H}$ is codense and so lemma 1.1 $c^*_\mu(i_\mu(A)) = c_\mu(i_\mu(A))$. Again since $\mathscr{H}$ is completely μ-codense, by lemma 1.2 ,$\mathscr{H} \subset \mathscr{N}$ and so $A^*(\mathscr{N}) \subset A^*(\mathscr{H})$. Since $A^*(\mathscr{N}) = c_\mu(i_\mu(c_\mu(A)))$, we have $c_\mu(i_\mu(c_\mu(A))) \subset A^* \subset c^*_\mu(A)$ and so $i_\mu(c_\mu(i_\mu(c_\mu(A)))) \subset i_\mu(c^*_\mu(A))$ which implies that $i_\mu(c_\mu(A)) \subset i_\mu(c^*_\mu(A))$. Now $A \subset i_\mu(c_\mu(A)) \cup c_\mu(i_\mu(A)) \subset i_\mu(c^*_\mu(A)) \cup c^*_\mu(i_\mu(A))$ and so A is b-$\mathscr{H}$-open.

3. $B_\mathscr{H}$-sets, $C_\mathscr{H}$-sets and $S_\mathscr{H}$-sets

Definition 3.1. A subset A of a hereditary generalized topological space $(X, \mu, \mathscr{H})$ is said to be

a) t-$\mathscr{H}$-set, if $i_\mu(c^*_\mu(A)) = i_\mu(A)$

b) t^*-$\mathscr{H}$-set, if $i_\mu(c^*_\mu(i_\mu(A))) = i_\mu(A)$

c) S-$\mathscr{H}$-set, if $c^*_\mu(i_\mu(A)) = i_\mu(A)$.

Definition 3.2. A subset A of a hereditary generalized topological space $(X, \mu, \mathscr{H})$ is said to be

a) $B_\mathscr{H}$-set, if $A = U \cap V$, where $U \in \mu$ and V is t-$\mathscr{H}$-set

b) $C_\mathscr{H}$-set, if $A = U \cap V$, where $U \in \mu$ and V is an t^*-$\mathscr{H}$-set

c) $S_\mathscr{H}$-set, if $A = U \cap V$, where $U \in \mu$ and V is an S-$\mathscr{H}$-set.

Proposition 3.3. Let $(X, \mu, \mathscr{H})$ be a strong hereditary generalized topological space and $A \subset X$. Then the following holds:

a) If A is μ-t-set, then A is t-$\mathscr{H}$-set.

b) If $A = A^*$, then A is t-$\mathscr{H}$-set.

c) If A is t-$\mathscr{H}$-set, then A is $B_\mathscr{H}$-set.

Proof. *a*) Let A be a μ-t-set. Then $i_\mu(c^*_\mu(A)) = i_\mu(A^* \cup A) \subset i_\mu(c_\mu(A) \cup A) = i_\mu(c_\mu(A))$= $i_\mu(A)$. Obiously $i_\mu(c^*_\mu(A)) \supset i_\mu(A)$. Therefore $i_\mu(c^*_\mu(A)) = i_\mu(A)$. Hence A is t-$\mathscr{H}$-set.

b) The proof is obvious.

c) Let A be a t-$\mathscr{H}$-set. If we take $U = X \in \mu$, then $A = U \cap A$. Hence A is $B_{\mathscr{H}}$-set.

Remark 3.4. The following examples show that the converse of Proposition 3.3 need not be true.

Example 3.5. *Let* $X = \{a,b,c,d\}$, $\mu = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X\}$ *and* $\mathscr{H} = \{\emptyset, \{a\}, \{b\}\}$. *Then* $A = \{a,b\}$ *is* t-$\mathscr{H}$-*set but not* μ-t-*set and* $A = A^*$.

Example 3.6. *Let* $X = \{a,b,c,d\}$, $\mu = \{\emptyset, \{a\}, \{c\}, \{a,b,c\}, \{b,c,d\}, X\}$ *and* $\mathscr{H} = \{\emptyset, \{a\}, \{b\}\}$. *Then* $A = \{b,c\}$ *is B-*$\mathscr{H}$*-set but not* t-$\mathscr{H}$-*set.*

Proposition 3.7. *Let* $(X, \mu, \mathscr{H})$ *be a hereditary generalized topological space and* $A \subset X$. *Then the following holds:*

a) *If A is a* μ-t^*-*set, then A is an* t^*-$\mathscr{H}$-*set.*

b) *If A is a* t-$\mathscr{H}$-*set, then A is an* t^*-$\mathscr{H}$-*set.*

c) *If A is a* t^*-$\mathscr{H}$-*set, then A is a* $C_{\mathscr{H}}$-*set.*

Proof. *a*) Let A be μ-t^*-set. Then $i_\mu(c^*_\mu(i_\mu(A))) \subset i_\mu(c_\mu(i_\mu(A)) \cup i_\mu(A)) \subset i_\mu(c_\mu(i_\mu(A))) = i_\mu(A)$. clearly $c^*_\mu(i_\mu(A)) \supset i_\mu$(A). Hence $i_\mu(c^*_\mu(i_\mu(A))) = i_\mu(A)$. Thus A is t^*-$\mathscr{H}$-set.

b) Let A be t-$\mathscr{H}$-set. Then $i_\mu(A) = i_\mu(c^*_\mu(A)) \supset i_\mu(c^*_\mu(i_\mu(A))) \supset i_\mu(A)$. Hence A is t^*-$\mathscr{H}$-set.

c) The proof is obvious.

Remark 3.8. The following examples show that the converse of Proposition 3.7, need not be true.

Example 3.9. *Let* $X = \{a,b,c,d\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{b,c,d\}, X\}$ *and* $\mathscr{H} = \{\emptyset, \{a\}, \{b\}\}$. *Then* $A = \{a,b\}$ *is* t^*-$\mathscr{H}$-*set but not* μ-t^*-*set and* t-$\mathscr{H}$-*set.*

Example 3.10. *Let* $X = \{a,b,c,d\}$, $\mu = \{\emptyset, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}, X\}$ *and* $\mathscr{H} = \{\emptyset, \{a\}, \{b\}\}$. *Then* $A = \{a,c\}$ *is* $C_{\mathscr{H}}$-*set but not* t^*-$\mathscr{H}$-*set.*

Theorem 3.11. *Let* $(X, \mu, \mathscr{H})$ *be a strong hereditary generalized topological space. For a subset A of* $(X, \mu, \mathscr{H})$, *then the following conditions are equivalent:*

a) A is μ-open.

b) A is π-$\mathscr{H}$-open and $B_{\mathscr{H}}$-set.

c) A is α-$\mathscr{H}$-open and a $C_{\mathscr{H}}$-set.

d) A is σ-$\mathscr{H}$-open and a $S_{\mathscr{H}}$-set.

Proof. *a)* Let A be μ-open. Now $i_\mu(c^*_\mu(A)) = c^*_\mu(A) \supset A$. Hence A is π-$\mathscr{H}$-open.

$(a) \Rightarrow (b)$,$(a) \Rightarrow (c)$ and $(a) \Rightarrow (d)$ are obvious, because X is a t-$\mathscr{H}$ set, t^*-$\mathscr{H}$-set and $S_{\mathscr{H}}$-set.

$(b) \Rightarrow (a)$ Since A is π-$\mathscr{H}$-open, $A \subset i_\mu(c^*_\mu(A)) = i_\mu(c^*_\mu(U \cap V))$, where V is t-$\mathscr{H}$-set. Hence A is μ-open.

Remark 3.12. The following examples show that the notations of π-$\mathscr{H}$-open (resp. α-$\mathscr{H}$-open set, σ-$\mathscr{H}$-open set) and $B_{\mathscr{H}}$-set (resp. $C_{\mathscr{H}}$-set, $S_{\mathscr{H}}$-set) are independent.

Example 3.13. *Let $X = \{a,b,c,d\}$, μ=$\{\emptyset, \{c\}, \{a,b\}, \{b,c,d\}, \{a,b,c\}, X\}$ and $\mathscr{H}$=$\{\emptyset, \{d\}\}$. Then*

1. $A = \{a,d\}$ is $B_{\mathscr{H}}$-set but not π-$\mathscr{H}$-open.

2. $A = \{a,c\}$ is π-$\mathscr{H}$-open but not $B_{\mathscr{H}}$-set.

Example 3.14. *Let $X = \{a,b,c,d\}$, μ=$\{\emptyset, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}, X\}$ and $\mathscr{H}$=$\{\emptyset, \{a\}, \{b\}\}$. Then A=$\{a,b,d\}$ is $C_{\mathscr{H}}$-set but not α-$\mathscr{H}$-open.*

Example 3.15. *Let $X = \{a,b,c,d\}$, μ=$\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{b,c,d\}, X\}$ and $\mathscr{H} = \{\emptyset, \{a\}, \{b\}\}$. Then A=$\{b,c,d\}$ is α-$\mathscr{H}$-open but not $C_{\mathscr{H}}$-set.*

Example 3.16. *Let $X = \{a,b,c,d\}$, μ=$\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{b,c,d\}, X\}$ and $\mathscr{H}$=$\{\emptyset, \{a\}, \{b\}\}$. Then*

1. $A = \{c\}$ is σ-$\mathscr{H}$-open but not $S_{\mathscr{H}}$-set.

2. $A = \{a,d\}$ is $S_{\mathscr{H}}$-set but not σ-$\mathscr{H}$-open set.

Theorem 3.17. *Let $(X, \mu, \mathscr{H})$ be a hereditary generalized topological space and $A \subset X$. If A is μ^*-closed and σ-$\mathscr{H}$-open, then A is t^*-$\mathscr{H}$-set.*

Proof: *Let A be σ-$\mathscr{H}$-open. Then $A \subset c^*_\mu(i_\mu(A))$. Now $i_\mu(c^*_\mu(A)) \subset c^*_\mu(A) \subset A$. Hence $A = c^*_\mu(i_\mu(A))$. Thus $i_\mu(A) = i_\mu(c^*_\mu(i_\mu(A)))$. Hence A is t^*-$\mathscr{H}$-open.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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