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ON A $M^{[X]}/(G_1,G_2)/1$ QUEUE WITH THIRD-STAGE OPTIONAL SERVICE AND DETERMINISTIC SERVER VACATIONS

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Abstract: We study a single server queueing system with Poisson arrivals in batches of variable size, assuming that customers in a batch are pre-ordered for the purpose of service. The server provides two stages of general heterogeneous service to all customers one by one on the basis of first come, first severed rule. A customer whose two stages of service are complete, may decide to take the third stage optional service with a certain probability or may just leave the system without taking the third stage service. Every time when a customer's required number of services (two or three as the case may be) is complete, the server has the option to take a vacation with a certain probability or continue staying in the system to serve the other customers. We further assume that whenever the server takes a vacation, it is a vacation of fixed length'd'. We find steady state probability generating function of number of customers in an explicit form, the steady state probabilities of various states of the system, the average queue size, the average system size, and the average waiting time in the queue as well as the average waiting time in the system.

Keywords: Batch arrivals, two stages of service, third stage optional service, deterministic server vacations, steady state
2010 AMS Subject Classification: 37K40.

1. Introduction

A large number of reserachers including Borthakur and Choudhury [1], Choudhury [2, 3], Madan and Choudhury [15], Gaver [5], Kielson and Servi [6], Lee and Srinivan [8], Rosenberg and Yechialli [17], Servi [18,19] and Tegham [23] have studied a single arrival or batch arrival queueing systems with server vacations. Most of these studies assume Bernoulli server vacations or modified Bernoulli server vacations where, after compeltion of each service, the server may take a avacation of random length of time with a certain probability or may continue serving the

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customers or remaining idle but available in the system even if there is no customer in the queue. Among the authors who studied queueing systems with vacation policies other than the Bernoulli type vacations, we mention Shanthikumar [20] who studied generalized vacations, Madan & Abu-Rub [12,13,14] who considered exhaustiev vacations and Madan [9] considered a priority queueing system with exhaustive service in which the server cannot take a vacation till all priority units present in the system are served. Recently, A. Krishnamoorthy and C. Sreeniwan [7] and Tao et al [22] studied queueing system with working vacations wherein they assume that the server is on vacation but keeps working in the system at a lower rate. Further, most of the above-mentined authors assumed that the server takes a single vacation and some, e.g. Choudhury and Madan [4] studied a queueing system in which server may take multiple vacations. Further, majority of authors who studied vacation queues assume that the server takess a vacation of random length. However, in many real life situations, the server may take a break or a vacation of fixed lenghth as it happens in a factory, a bank, a railway station and a post office etc.. In order to minimize uncertainty of availability of a server, a fixed length vacation is more realistic in many queueing situations. Madan [10,16] studied queueing systems with deterministic vacations. Further, Madan [11] introduced the idea of a second optional service in a single server queue without server vacations. In the prersent paper we cosider a queueing system which provides two stages of general service followed by a third stage optional service with deterministic server vacations. We generalizes results obtained by Madan [10], Madan [11] and Madan [16].

2. The Mathematical Model

Customers (units) arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda\, c_i dt\,(i = 1,2,3,...)$ be the first order probability that a batch of i customers arrives at system during the short interval of time $(t, t+dt]$, where $0 \le c_i \le 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. There is a single server who provides a two-stage general heterogeneous service to customers one-by-one on a first come, first served (FCFS) basis. As soon as the stage two service of a customer is complete, the customer may opt for third-stage optional service with probability p or else may leave the system with probability $1-p$. We assume that the service time random variable S_j of $j\,th$ stage service follows a general probability law with the

distribution function $B_j(s_j)$, and the probability density function $b_j(s_j)$ and let $E(S_j^k)$ be the $k\,th$ moment ($k=1,2,...$) of the $j\,th$ service time, $j=1,2,3.$

Let $\mu_j(x)$ be the conditional probability of completion of type j service during the interval $(x, x+dx]$, given that the elapsed time is x, so that

$$\mu_j(x)=\frac{b_j(x)}{1-B_j(x)},\ j=1,2,3,, \tag{2.1}$$

and therefore,

$$b_j(s_j)=\mu_j(s_j)e^{-\int_0^{s_j}\mu_j(x)dx}\quad j=1,2.,3\,. \tag{2.2}$$

As soon as the number of services required by a customer i.e. either (G_1,G_2) or (G_1,G_2) followed by G_3 are complete, the server may decide to take a vacation for a constant duration $d\,(>0)$ with probability δ or else may continue to be in the system, either providing service to the next customer, if any or else remains idle and waits for the next batch of customers to arrive.

We further assume that whenever the server takes a vacation, it is always a single vacation. In other words, on completion of a vacation, the server must be back to the system even if there is no customer present in the system. Such a system with modified Bernoulli schedules has been recently studied by Madan and Abu Rub [14].

Remark: Here we may remark that this assumption is unlike the assumptions of several other authors who assume that on completion of a vacation, if the server finds no customers in the queue, then he must take another vacation.

Finally, it is assumed that the inter-arrival times of the customers, the service times of each kind of service and vacation times of the server, all these stochastic processes involved in the system are independent of each other.

3. Definitions, Notations and Equations Governing the system.

Assuming that the steady state exists let $P_{n,j}(x)$ denote the steady state probability that there are $n\,(\geq 0)$ customers in the queue excluding one customer in type j service, $j=1,2,3$ and the elapsed service time of this customer is x. Accordingly, $P_{n,j}=\int_0^{\infty}P_{n,j}(x)dx$ denotes the

corresponding steady state probability irrespective of the elapsed service time x. Next, we define V_n as a steady state probability that there are $n(\geq 0)$ customers in the queue and the server is on vacation. Finally, let *E* denote the steady state probability that the system is empty that is there is no customer either in queue or in service and the server is idle but available in the system. We further assume that K_r is the probability of r arrivals during the vacation period and therefore,

$$K_r = \frac{\exp(\lambda d)(\lambda d)^r}{r!}, \quad r = 0,1,2,.... \tag{3.1}$$

In addition, we define the following probability generating functions (PGFs):

$$P_j(x,z) = \sum_{n=0}^{\infty} P_{n,j}(x) z^n \ , P_j(z) = \sum_{n=0}^{\infty} P_{n,j} z^n \ , j = 1,2, \tag{3.2.a}$$

$$V(z) = \sum_{n=0}^{\infty} V_n \, z^n \ , \tag{3.2b}$$

$$C(z) = \sum_{i=1}^{\infty} c_i z^i \ , \tag{3.2c}$$

$$K(z) = \sum_{n=0}^{\infty} K_n \, z^n = \sum_{n=0}^{\infty} \frac{\exp(-\lambda d)(\lambda d)^n}{n!} \, z^n = \exp\,[-\lambda d(1-z)], \quad |z| \leq 1. \tag{3.2d}$$

Usual probability reasoning based on assumptions of our mathematical model, we obtain the following forward steady state equations for the system:

$$\frac{d}{dx} P_{n,1}(x) + \big(\lambda + \mu_1(x)\big) P_{n,1}(x) = \lambda \sum_{i=1}^{n} c_i P_{n-i,1}(x) \ , n \geq 1, \tag{3.3}$$

$$\frac{d}{dx} P_{0,1}(x) + \big(\lambda + \mu_1(x)\big) P_{0,1}(x) = 0, \tag{3.4}$$

$$\frac{d}{dx} P_{n,2}(x) + \big(\lambda + \mu_2(x)\big) P_{n,2}(x) = \lambda \sum_{i=1}^{n} c_i P_{n-i,2}(x) \ , n \geq 1, \tag{3.5}$$

$$\frac{d}{dx} P_{0,2}(x) + \big(\lambda + \mu_2(x)\big) P_{0,2}(x) = 0, \tag{3.6}$$

$$\frac{d}{dx} P_{n,3}(x) + \big(\lambda + \mu_3(x)\big) P_{n,3}(x) = \lambda \sum_{i=1}^{n} c_i P_{n-i,3}(x) \ , n \geq 1, \tag{3.7}$$

$$\frac{d}{dx} P_{0,3}(x) + \big(\lambda + \mu_3(x)\big) P_{0,3}(x) = 0, \tag{3.8}$$

$$V_n = \delta(1-p)\int_0^\infty P_{n,2}(x)\mu_2(x)dx + \delta\int_0^\infty P_{n,3}(x)\mu_3(x)dx, \quad n \geq 0, \tag{3.9}$$

$$\lambda E = (1-\delta)(1-p)\int_0^\infty P_{0,2}(x)\mu_2(x)dx + (1-\delta)\int_0^\infty P_{0,3}(x)\mu_3(x)dx + V_0\, K_0 \tag{3.10}$$

The above equations have to be solved subject to the following boundary conditions:

$$P_{n,1}(0) = (1-p)(1-\delta)\int_0^\infty P_{n+1,2}(x)\mu_2(x)dx + (1-\delta)\int_0^\infty P_{n+1,3}(x)\mu_3(x)dx$$

$$V_0K_{n+1} + V_1K_n + \ldots + V_{n+1}K_0 + \lambda c_{n+1}E, \quad n \geq 1, \tag{3.11}$$

$$P_{0,1}(0) = (1-p)(1-\delta)\int_0^\infty P_{1,2}(x)\mu_2(x)dx + (1-\delta)\int_0^\infty P_{1,3}(x)\mu_3(x)dx$$

$$V_0K_1 + V_1K_0 + \lambda c_1E\ , \tag{3.12}$$

$$P_{n,2}(0) = \int_0^\infty P_{n,1}(x)\mu_1(x)dx, \quad n \geq 0, \tag{3.13}$$

$$P_{n,3}(0) = p(1-\delta)\int_0^\infty P_{n,2}(x)\mu_2(x)dx, \ n \geq 0. \tag{3.14}$$

4. Steady-state probability generating functions

We multiply equation (3.3) by z^n, take summation over n. add (3.4) to the sum, use (3.1 a) and simplify. Thus we obtain

$$\frac{d}{dz}P_1(x,z) + \big(\lambda - \lambda C(z) + \mu(x)\big)P_1(x,z) = 0. \tag{3.15}$$

Similar operations on (3.5) & (3.6), (3.7) & (3.8), (3.9) alone, (3.11) & (3.12), (3.13) alone, (3.14) alone and using (3.2 a) to (3.2d), we obtain

$$\frac{d}{dz}P_2(x,z) + \big(\lambda - \lambda C(z) + \mu(x)\big)P_2(x,z) = 0, \tag{3.16}$$

$$\frac{d}{dz}P_3(x,z) + \big(\lambda - \lambda C(z) + \mu(x)\big)P_3(x,z) = 0, \tag{3.17}$$

$$V(z) = \delta(1-p)\int_0^\infty P_2(x,z)\mu_2(x)dx + \delta\int_0^\infty P_3(x,z)\mu_3(x)dx, \ , \tag{3.18}$$

$$zP_1(0,z) = (1-p)(1-\delta)\int_0^\infty P_2(x,z)\mu_2(x)dx + (1-\delta)\int_0^\infty P_3(x,z)\mu_3(x)dx$$

$$+ K(z)V(z) + \lambda C(z)E - (1-p)(1-\delta)\int_0^\infty P_{0,2}(x)\mu_2(x)dx$$

$$-(1-\delta)\int_0^\infty P_{0,3}(x)\mu_3(x)dx - V_0K_0\,, \tag{3.19}$$

$$P_2(0,z) = \int_0^\infty P_1(x,z)\mu_1(x)dx, \tag{3.20}$$

$$P_3(0,z) = p(1-\delta)\int_0^\infty P_2(x,z)\mu_2(x)dx\,. \tag{3.21}$$

Now, using (3.10) and (3.1), we can re-write equation (3.19) as

$$zP_1(0,z) = (1-p)(1-\delta)\int_0^\infty P_2(x,z)\mu_2(x)dx + (1-\delta)\int_0^\infty P_3(x,z)\mu_3(x)dx$$

$$+ K(z)V(z) + \lambda\big(C(z)-1\big)E \tag{3.22}$$

Next, we integrate equations (3.15), (3.16) and (3.17) between the limits 0 and x and obtain

$$P_1(x,z) = P_1(0,z)\exp\left[-\big(\lambda-\lambda C(z)\big)x - \int_0^\infty \mu_1(t)dt\right], \tag{3.23}$$

$$P_2(x,z) = P_2(0,z)\exp\left[-\big(\lambda-\lambda C(z)\big)x - \int_0^\infty \mu_2(t)dt\right], \tag{3.24}$$

$$P_3(x,z) = P_3(0,z)\exp\left[-\big(\lambda-\lambda C(z)\big)x - \int_0^\infty \mu_3(t)dt\right], \tag{3.25}$$

where $P_1(0,z)$, $P_2(0,z)$ and $P_3(0,z)$ have been obtained above.

We again integrate (3.23), (3.24) and (3.25) with respect to x and obtain

$$P_1(z) = P_1(0,z)\left(\frac{1-\overline{B}_1\big(\lambda-\lambda C(z)\big)}{\lambda-\lambda C(z)}\right), \tag{3.26}$$

$$P_2(z) = P_2(0,z)\left(\frac{1-\overline{B}_2\big(\lambda-\lambda C(z)\big)}{\lambda-\lambda C(z)}\right), \tag{3.27}$$

$$P_3(z) = P_3(0,z)\left(\frac{1-\overline{B}_3(\lambda-\lambda C(z))}{\lambda-\lambda C(z)}\right), \tag{3.28}$$

where $\overline{B}_j(\lambda-\lambda C(z)) = \int_0^\infty \exp\left[-(\lambda-\lambda C(z))x\right] dB_j(x)$ the Laplace-Stieltjes transform of the j th stage service time, $j = 1,2,3.$

Now, we have to determine the integrals $\int_0^\infty P_1(x,z)\mu_1(x)dx$, $\int_0^\infty P_2(x,z)\mu_2(x)dx$ and $\int_0^\infty P_3(x,z)\mu_3(x)dx$ which appear in the right hand side of equations (3.18), (3.20), (3.21) and (3.22). For this purpose, we multiply equations (3.27), (3.28) and (3.29) by $\mu_1(x)$, $\mu_2(x)$ and $\mu_3(x)$ respectively and integrate them with respect to x and use (2.2). Thus we obtain

$$\int_0^\infty P_1(x,z)\mu_1(x)dx = P_1(0,z)\,\overline{B}_1(\lambda-\lambda C(z)), \tag{3.29}$$

$$\int_0^\infty P_2(x,z)\mu_1(x)dx = P_2(0,z)\,\overline{B}_2(\lambda-\lambda C(z)), \tag{3.30}$$

$$\int_0^\infty P_3(x,z)\mu_1(x)dx = P_3(0,z)\,\overline{B}_3(\lambda-\lambda C(z)). \tag{3.31}$$

Using (3.29), (3.30) and (3.31) in equations (3.18), (3.20), (3.21) and (3.22), we obtain

$$V(z) = \delta(1-p)P_2(0,z)\,\overline{B}_2(\lambda-\lambda C(z)) + \delta P_3(0,z)\overline{B}_3(\lambda-\lambda C(z)), \tag{3.32}$$

$$P_2(0,z) = P_1(0,z)\overline{B}_1(\lambda-\lambda C(z)), \tag{3.33}$$

$$P_3(0,z) = p\,P_2(0,z)\overline{B}_2(\lambda-\lambda C(z)), \tag{3.34}$$

$$\begin{aligned} zP_1(0,z) &= (1-p)(1-\delta)P_2(0,z)\overline{B}_2(\lambda-\lambda C(z)) + (1-\delta)P_3(0,z)\overline{B}_3(\lambda-\lambda C(z)) \\ &\quad + K(z)V(z) + \lambda E(C(z)-1). \end{aligned} \tag{3.35}$$

Next, we solve equations (3.32) to (3.35) recursively for $P_1(0,z)$, $P_2(0,z)$, $P_3(0,z)$ and $V(z)$ get

$$P_1(0,z) = \frac{\lambda E(C(z)-1)}{D(z)}, \tag{3.36}$$

$$P_2(0,z) = \frac{\lambda E(C(z)-1)\overline{B}_1(\lambda-\lambda C(z))}{D(z)}, \tag{3.37}$$

$$P_3(0,z)=\frac{p(1-\delta)\lambda E\left(C(z)-1\right)\overline{B}_1\left(\lambda-\lambda C(z)\right)\overline{B}_2\left(\lambda-\lambda C(z)\right)}{D(z)}, \tag{3.38}$$

$$V(z)=\frac{\left[\begin{array}{l}\lambda E\left(C(z)-1\right)\delta(1-p)\overline{B}_1\left(\lambda-\lambda C(z)\right)\overline{B}_2\left(\lambda-\lambda C(z)\right)\\ +\lambda E\left(C(z)-1\right)\delta p\overline{B}_1\left(\lambda-\lambda C(z)\right)\overline{B}_2\left(\lambda-\lambda C(z)\right)\overline{B}_3\left(\lambda-\lambda C(z)\right)\end{array}\right]}{D(z)}, \tag{3.39}$$

where

$$\begin{aligned}D(z)=z&-(1-p)(1-\delta)\overline{B}_1\left(\lambda-\lambda C(z)\right)\overline{B}_2\left(\lambda-\lambda C(z)\right)\\&-\delta(1-p)K(z)\overline{B}_1\left(\lambda-\lambda C(z)\right)\overline{B}_2\left(\lambda-\lambda C(z)\right)\\&-\delta\, p\,K(z)\overline{B}_1\left(\lambda-\lambda C(z)\right)\overline{B}_2\left(\lambda-\lambda C(z)\right)\overline{B}_3\left(\lambda-\lambda C(z)\right)\\&-p(1-\delta)\overline{B}_1\left(\lambda-\lambda C(z)\right)\overline{B}_2\left(\lambda-\lambda C(z)\right)\overline{B}_3\left(\lambda-\lambda C(z)\right).\end{aligned} \tag{3.40}$$

Next, we utilize (3.36), (3.37) and (3.38) into (3.26), (3.27) and (3.28) respectively and on simplifying we obtain

$$P_1(z)=\frac{\left(\overline{B}_1\left(\lambda-\lambda C(z)\right)-1\right)E}{D(z)}\ , \tag{3.41}$$

$$P_2(z)=\frac{\overline{B}_1\left(\lambda-\lambda C(z)\right)\left(\overline{B}_2\left(\lambda-\lambda C(z)\right)-1\right)E}{D(z)}\ , \tag{3.42}$$

$$P_3(z)=\frac{p(1-\delta)\,\overline{B}_1\left(\lambda-\lambda C(z)\right)\overline{B}_2\left(\lambda-\lambda C(z)\right)\left(\overline{B}_3\left(\lambda-\lambda C(z)\right)-1\right)E}{D(z)}\ , \tag{3.43}$$

Next, we determine

$$\begin{aligned}V(1)&=\lim_{z\to 1}V(z)\\&=\lim_{z\to 1}\frac{\left[\begin{array}{l}\lambda E\left(C(z)-1\right)\delta(1-p)\overline{B}_1\left(\lambda-\lambda C(z)\right)\overline{B}_2\left(\lambda-\lambda C(z)\right)\\ +\lambda E\left(C(z)-1\right)\delta p\overline{B}_1\left(\lambda-\lambda C(z)\right)\overline{B}_2\left(\lambda-\lambda C(z)\right)\overline{B}_3\left(\lambda-\lambda C(z)\right)\end{array}\right]}{D(z)}\\&=\frac{\lambda\delta\,E(I)\,E}{1-\lambda E(I)\left(E(B_1)+E(B_2)+\lambda pE(B_3)\right)-\delta\lambda d},\end{aligned} \tag{3.44}$$

$$\begin{aligned}P_1(1)&=\lim_{z\to 1}P_1(z)=\lim_{z\to 1}\frac{\left(\overline{B}_1\left(\lambda-\lambda C(z)\right)-1\right)E}{D(z)}\\&=\frac{\lambda\,E(I)\,E(B_1)E}{1-\lambda E(I)\left(E(B_1)+E(B_2)+\lambda\,pE(B_3)\right)-\delta\lambda d},\end{aligned} \tag{3.45}$$

$$P_2(1) = \lim_{z\to 1} P_2(z) = \lim_{z\to 1} \frac{\overline{B}_1(\lambda-\lambda C(z))\left(\overline{B}_2(\lambda-\lambda C(z))-1\right)E}{D(z)}$$

$$= \frac{\lambda E(I)E(B_2)E}{1-\lambda E(I)\left(E(B_1)+E(B_2)+\lambda pE(B_3)\right)-\delta\lambda d}, \tag{3.46}$$

$$P_3(1) = \lim_{z\to 1} P_3(z)$$

$$= \lim_{z\to 1} \frac{p(1-\delta)\overline{B}_1(\lambda-\lambda C(z))\overline{B}_2(\lambda-\lambda C(z))\left(\overline{B}_3(\lambda-\lambda C(z))-1\right)E}{D(z)}$$

$$= \frac{\lambda p(1-\delta)E(I)E(B_3)E}{1-\lambda E(I)\left(E(B_1)+E(B_2)+\lambda pE(B_3)\right)-\delta\lambda d}, \tag{3.47}$$

where $E(I)$ is the average batch size and $E(B_j)$, $j=1,2,3$ is the mean service time of *jth* stage service.

In order to determine the unknown probability E appearing in the numerators of equations ((3.44) to (3.47), we put $z=1$ in these equations and use the normalizing condition

$$E + P_1(1) + P_2(1) + P_3(1) + V(1) = 1. \tag{3.48}$$

On simplify (3.48) we obtain

$$E = \frac{1-\lambda E(I)\left(E(B_1)+E(B_2)+pE(B_3)\right)-\delta\lambda d}{1+\lambda E(I)\delta\left(1-pE(B_3)\right)-\delta\lambda d}\ . \tag{3.49}$$

We note that (3.49) give the probability that the server is idle but available in the system. Also this gives us

$$\lambda E(I)\left(E(B_1)+E(B_2)+pE(B3)\right)+\delta\lambda d<1, \tag{3.50}$$

which is the stability condition under which the steady state exists.

Next, we substitute the value of E found in (3.49) in equations (3.39), (3.41), (3.42) and (3.43) and thus $V(z)$, $P_1(z)$, $P_2(z)$ and $P_3(z)$ are completely determined.

5. Steady-state average number of customers and average waiting time in the queue as well as in the system

We define $P_Q(z)$ as the steady state probability generating function for the number of customers in the queue so that adding (3.39), (3.41), (3.42) and (3.43) we get

$$P_Q(z) = P_1(z) + P_2(z) + P_3(z) + V(z) = \frac{N(z)}{D(z)} \tag{5.1}$$

Let L_q denote the steady state average number of customers in the queue. Then

$L_Q = \frac{d}{dz} P_Q(z)|_{z=1}$. However, since $P_Q(z) = \frac{0}{0}$ at z =1, we use double differentiation and obtain

$$L_Q = \lim_{z\to 1} \frac{d}{dz} P_Q(z) = \lim_{z\to 1} \frac{D'(z)N''(z) - N'(z)D''(z)}{2\left[D'(z)\right]^2}$$

$$= \frac{D'(1)N''(1) - N'(1)D''(1)}{2\left[D'(1)\right]^2}, \qquad (5.2)$$

where primes mean derivatives with respect to z and after a lot of algebra and simplification, we obtain

$$L_Q = \frac{\left(\lambda E(I)\right)^2 \left[E(B_1^2) + E(B_2^2) + pE(B_3^2) + 2\left(E(B_1) + E(B_2) + pE(B_3)\right)\right]}{2(1-\rho)}$$

$$+ \frac{2\delta\lambda d\, \lambda E(I)\left[E(B_1) + E(B_2) + pE(B_3)\right] + \delta\lambda^2 d}{2(1-\rho)} + \frac{\rho E(X_I)}{1-\rho}, \qquad (5.3)$$

where $\rho = \lambda E(I)\left(E(B_1) + E(B_2) + pE(B3)\right) + \delta\lambda d$, $E(B_j), j = 1, 2, 3$ is the mean service time of jth stage service, $E(B_j^2), j = 1, 2, 3$ is the second moment of jth stage service time and $E(X_I) = \frac{E(I(I-1))}{2E(I)}$ is the mean residual batch size.

Note that using L_q obtained in (5.3) into Little's formulae we can obtain the following:

The steady state average number of customers in the system,

$$L = L_q + \rho, \qquad (5.4)$$

The steady state average waiting time in the queue,

$$W_q = \frac{L_q}{\lambda}, \qquad (5.5)$$

The steady state average waiting time in the system,

$$W = \frac{L}{\lambda}. \qquad (5.6)$$

6. Some particular cases

Case -1: Two stages of general service with general vacation times and no third service

In this case we let $p=0$ in the main results and derive results of Madan [ ] as a particular case.

Case -2: Two stages of general service with third optional general service and no server vacations

The results of this case can be obtained by letting $\delta=0$ in the main results.

Case -3: Two stages of general service with no third optional service and no server vacations

The results of this case can be obtained by letting $p=0$ and $\delta=0$ in the main results.

Conflict of Interests

The authors declare that there is no conflict of interests.

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